# Digital Signal Processing

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# INTRODUCTION

Signal:

- Parameter that depends upon another independent parameter.
- Any physical quantity that varies with time, space or any other independent variable or variables.
- Any physical quantity that carries information.

Independent variable may be time, spatial coordinates, position, pressure, depth, temp, etc.

Common signals:

Speech, Music — Audio signals

Video signals

Pictures, Photographs — Image signals

Voltage, Current — Electrical signals

Radio, Microwave, Satellite, Radar signals — Commn. Signals

Other signals

Velocity, Force — Mech. signals

Rates of reaction — Chem. signal

Earth vibration — Seismic signal.

- Signals may be one or multi dimensional

Eg.

Voltage v(t), Electrocardiogram ECG, Electroencephalogram EEG — Information bearing signals that evolve as function of a single independent variable namely time.

Image signal — Function of two independent variables. The variables are spatial coordinates.

# Signal Processing.

- Is the manipulation, enhancement or extraction of information from signals. It involves mathematical, technica & algorithmic procedure

- Is the action of changing one or more features (char of a signal according to a pre-determined requirements. Characteristics includes amplitude, phase & freq content of signal.

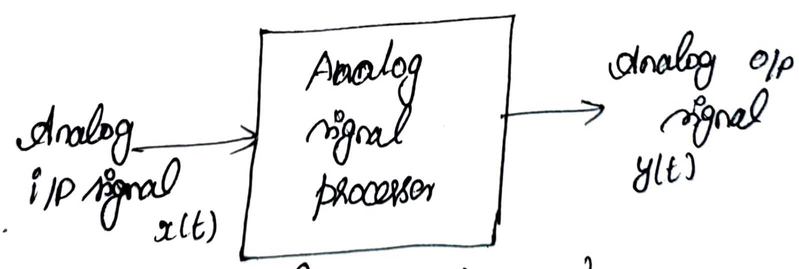
Two kinds:

Analog signal processing & Digital signal processing.

## Analog signal processing:

- Most of the natural signals are in analog for
- System that process the signals in analog form (continuous in time) is known as Analog signal processing system.

Eg: Filter, frequency analyzers, frequency multiplier, Attenuator, Modulator etc.

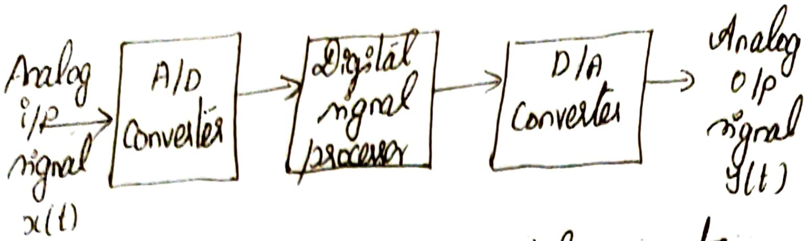



Analog signal processing system.

## Digital Signal processing:

- Digital signal processing provides an alternative method for processing the analog signal.
- process the signals that are discrete in time

# Basic elements of a digital signal processing system




- Eg: Large programmable digital computer or a small microprocessor programmed to perform the desired operation on the i/p signal. (or) a hardwired digital processor configured to perform a specified set of operations.
- Programmable machines provides the flexibility to change the signal processing operations through change in the s/w. whereas hardwired machines are difficult to reconfigure.

## Advantages of Digital over Analog signal processing

1. Flexibility in reconfiguration:
   - Digital programmable system allows the reconfigu by changing the pgm.
   - Analog system involves redesign of system.

2. Great accuracy:
   - Digital systems provides much better control of accuracy requirements which inturn result in specifying the accuracy requirements in A/D converter.
   - Tolerance in circuit components affects the accuracy of analog circuit.

3. Easy Storage

- Easily stored on magnetic tape or disk without deterioration beyond that introduce in the A/D converter.
- Signals become transportable & can be proce in a remote laboratory

4. Implementation of sophisticated algorithms:

- Allows for implementation of more sophistica signal processing algorithms in DSP
- But it is very difficult to perform precise math operations on signals in analog form

5. Cheaper:

- Digital signal processing is cheaper than its analog counterpart due to the fact that the digital h/w is cheaper & also the flexibility for modification.

Applications:

As a consequences of these advantages, DSP has been applied in broad range of disciplines

Eg:

- Speech processing
- Signal transmission on telephone channels
- Image processing & transmission
- Seismology & geophysics
- Oil exploration
- Detection of nuclear explosions
- Processing of signals received from outer space etc.

Limitations

- Speed of operation of A/D Converters
- Signals having extremely wide bandwidth require fast sampling rate A/D converter is fast digital signal processors. Hence there are analog signals with large bandwidths for which a digital signal processing approach is beyond the state of the art of digital h/w

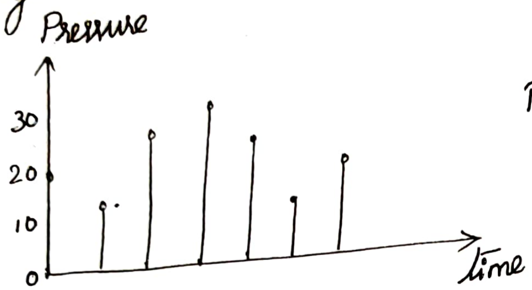
## Discrete-time Signals:

**Definition:**

# The signals that are defined at discrete instants of time are known as discrete-time signals. The discrete-time signals are continuous in amplitude & discrete in time. Denoted by $x(n)$.

# Is a function of independent variable that is an integer.

Eg:




Pressure Signal.

Note:

# Basic function (or) building block to describe more complex signals.

# Not defined at instants b/w two successive samples.

# Incorrect that $x(n)$ is equal to zero if $n$ is not integer. $x(n)$ is not defined for non integer values of $n$.

# Defined for every integer value of $n$ for $-\infty < n < \infty$.

By tradition, $x(n)$ is referred as $n^{th}$ sample of the signal even if the signal $x(n)$ is inherently discrete time. But if $x(n)$ was obtained from sampling an analog signal $x_a(t)$ Then $x(n) = x_a(nT)$, where $T$ is the sampling period.

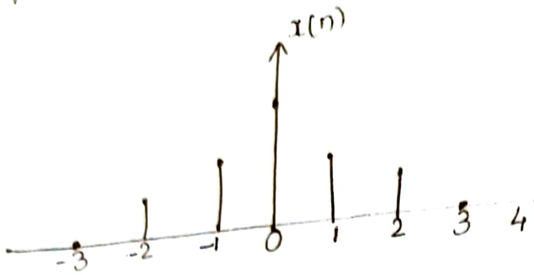
**Different types of representation for discrete-time signals:**

1. Functional representation

Eg. $$x(n) = \begin{cases} 1 & \text{for } n = 1, 3 \\ 4 & \text{for } n = 2 \\ 0 & \text{elsewhere} \end{cases}$$

2. Graphical representation

Eg.




3. Tabular representation:

Eg:

| n | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| x(n) | 0 | 1 | 2 | 3 | 1 | 2 | 0 |

4. Sequence representation

Eg:

Finite duration sequence with time origin n=0 indicated by the symbol ↑

$$x(n) = \{1, \underset{\uparrow}{3}, 4, 5, 6\}$$

Infinite duration sequence

$$x(n) = \{\dots 0, 2, \underset{\uparrow}{1}, -1, 3, -2, \dots\}$$

Finite duration sequence that satisfies the condition $x(n)=0$ for $n<0$.

$$x(n) = \{2, 4, 6, -2, -3, 1\}$$

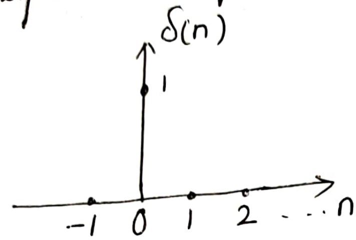
Elementary discrete - time signals:

1. Unit sample sequence

- Is a signal that is zero everywhere except at $n=0$ where its value is unity.
- Referred as "Unit Impulse."

$$\delta(n) = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

Graphical representation

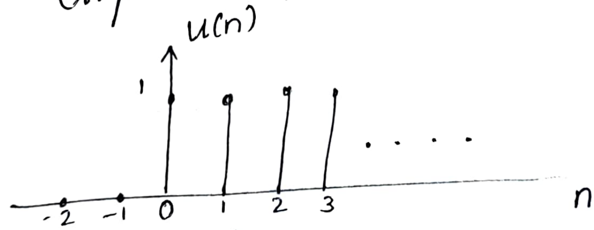



2. Unit step sequence:

- Defined as

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Graphical representation

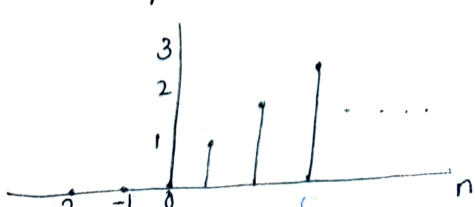



3. Unit ramp sequence:

- Defined as

$$U_r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Graphical representation



4. Exponential sequence:

- Is a sequence of the form

$$x(n) = a^n \text{ for all } n$$

Different types of discrete-time exponential signals

$a > 1$ - grows exponentially

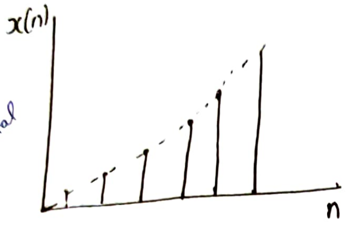
$a$ – ~~complex~~ real then $x(n)$ is real.

$x(t) = Ae^{at}$

$a = 0 \rightarrow$ A dc signal

$a > 1 \rightarrow$ Growing exponential

$a = +ve$




$0 < a < 1$ decays exponentially

$a$ – complex valued

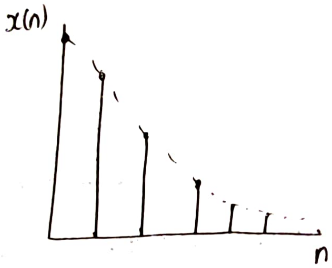
$a = re^{j\theta}$

$x(n) = r^n e^{j\theta n}$

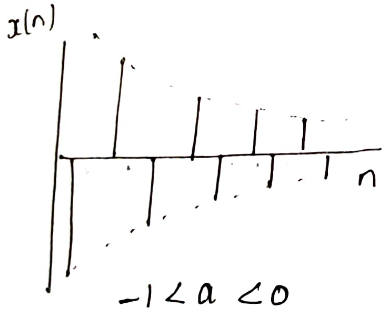
$= r^n(\cos\theta n + j\sin\theta n)$

$a < 1 \rightarrow$ decaying exponential

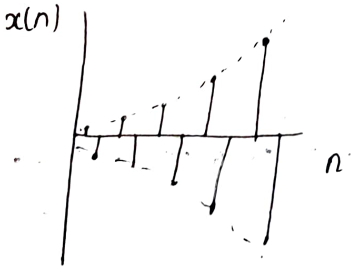
$a = -ve$




$a < 0$ takes alternating signs




$-1 < a < 0$




$a < -1$

5. Complex exponential sequence:

$a$ - Complex valued signal

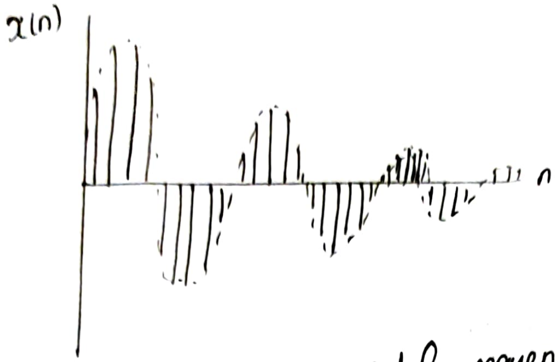
$$x(n) = a^n$$

where $a = re^{j\theta}$

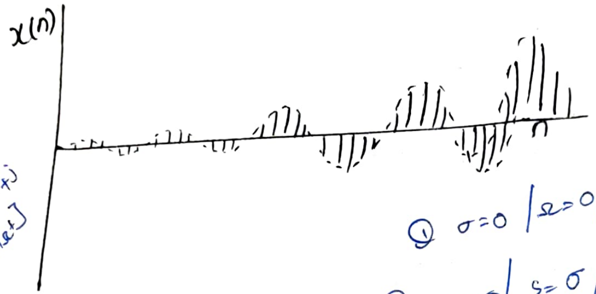
$$x(n) = r^n e^{j\theta n} = r^n(\cos\theta n + j\sin\theta n)$$

$|a| = 1$ Real & Imaginary parts of complex exponential s.. are sinusoidal

$|a| < 1$ Amplitude of the sinusoidal sequence decays exp..




$|a| > 1$ Amplitude of the sinusoidal sequence increase exponentia..




$x(t) = e^{st}$

$s = \sigma + j\Omega$

$= e^{(\sigma + j\Omega)t}$

$= e^{\sigma t} e^{j\Omega t}$

$e^{j\Omega t} = \cos\Omega t + j\sin\Omega t$

$x(t) = e^{\sigma t}[\cos\Omega t + j\sin\Omega t]$

① $\sigma = 0$ | $\Omega = 0$ | $x(t) = 1$

② $\Omega = 0$ | $s = \sigma$ | $x(t) = e^{\sigma t}$

6. Sinusoidal signals:

Is given by

$$x(n) = A\cos(\omega_0 n + \phi)$$

where $\omega_0$ - frequency - radians/sample

$\phi$ - phase - radians

Using Euler's identity

$$A\cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

since $|e^{j\omega_0 n}| = 1$, energy of the signal is infinite & av.. power of the signal is one

Classification of discrete time signals:

Mathematical methods employed in the analysis of discrete time signals & systems depend on the char of the signals.

According to a number of different characteristics, Discrete time signals are classified as

Energy signals & Power signals:

The energy E of a signal x(n) is defined as

$$E_x \text{ (or) } E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Magnitude - squared values of x(n) is used so it applies to complex - valued signals as well as real valued signals

Energy can be finite or infinite. If E is finite $(0 < E < \infty)$, then x(n) is a Energy Signal

Many signals that posses infinite energy, have a finite average power.

The average power of a discrete - time signal x(n) is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} E_N$$

where $E_N$ - energy of the signal x(n) over the finite interval $-N \le n \le N$ as

$$E_N = \sum_{n=-N}^{N} |x(n)|^2$$

Energy E is given by $E = \lim_{N \to \infty} E_N$

Note:

If $E$ - finite then $P = 0$ ∴ - Energy signal

On the otherhand

$E$ - infinite, $P$ may be either finite or infini

If $P$ - finite - Power signal.

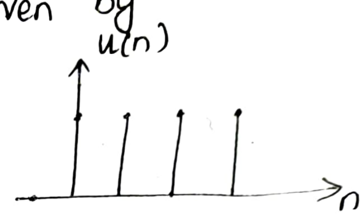
Example:

Determine the Power & Energy of the unit step seq.

sol

Unit step sequence is given by

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$




Energy

$$E_N = \sum_{n=-N}^{N} |x(n)|^2$$

$$= \sum_{n=0}^{N} u^2(n) = N+1$$

$$= u^2(0) + u^2(1) + \dots u^2(N)$$

$$E = \lim_{N \to \infty} E_N$$

$$= \lim_{N \to \infty} N+1 = \infty \text{ (Infinite)}$$

Power

$$P = \lim_{N \to \infty} \frac{1}{2N+1} E_N$$

$$= \lim_{N \to \infty} \frac{N+1}{2N+1}$$

$$= \lim_{N \to \infty} \frac{1 + 1/N}{2 + 1/N}$$

$$P = \frac{1}{2}$$

The unit step sequence is a power signal. It's average power is $1/2$. It has infinite Energy.

$$\frac{N+1}{2N+1} \quad \frac{\cancel{N}(1+1/N)}{\cancel{N}(2+1/N)}$$

Symmetric (Even) & Antisymmetric (odd) Signals:

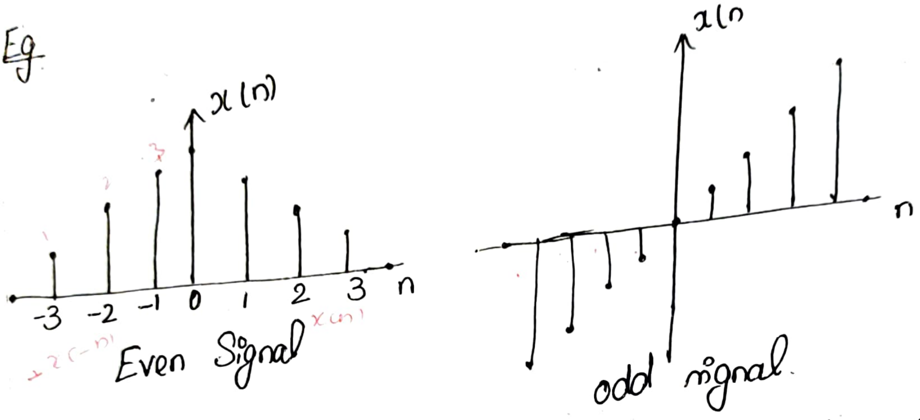
Real valued signal $x(n)$ is called symmetric (even) if $x(-n) = x(n)$ for all values of $n$.

On the otherhand, a signal $x(n)$ is called antisymmetric (odd) if $x(-n) = -x(n)$ for all values of $n$

Note:

If $x(n)$ – odd then $x(0) = 0$.

Eg.




Any arbitrary signal can be expressed as the sum of two signal components, one of which is even & the other odd. ie $x(n) = x_e(n) + x_o(n)$

The even signal component is formed by adding $x(n)$ to $x(-n)$ & dividing by 2

$$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$

This signal satisfies the symmetric condition.

Similarly

$$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

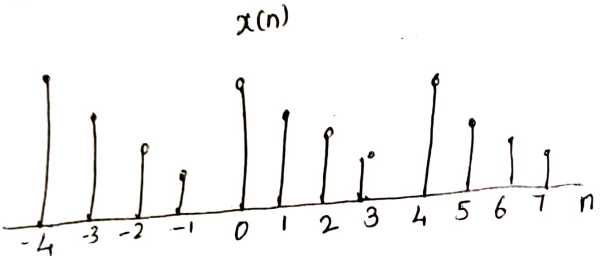
Periodic signal & Aperiodic signals:

Signal $x(n)$ is periodic with period $N$ $(N>0)$ if & only if $x(N+n) = x(n)$ for all $n$. The smallest value of $N$ for which

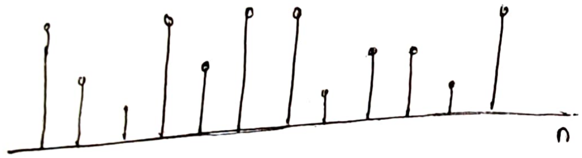
$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

satisfies is known as fundamental period. If there is no value of $N$ that satisfies the above condition is called Nonperiodic (or) Aperiodic.

Eg.




Periodic Signal




Aperiodic Signal

The discrete time sinusoidal signal of the form $x(n) = A \sin 2\pi f_0 n$ is periodic when $f_0$ is a rational number i.e $f_0 = \frac{k}{N}$ where $k$ & $N$ – integers.

i.e

$$x(n+N) = x(n)$$

$$x(n+N) = A \sin 2\pi f_0 (n+N)$$

$$= A \sin (2\pi f_0 n + 2\pi f_0 N)$$

The energy of a periodic signal $x(n)$ over a single period say over the interval $0 \le n \le N-1$ is finite if $x(n)$ takes on finite value over the period. However the energy the periodic signal for $-\infty \le n \le \infty$ is infinite.

The avg power of the periodic signal is finite & it is equal to the average power over a single period. Thus if $x(n)$ is a periodic signal with fundamental period $N$ & takes on finite values, its power is given by

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

ie periodic signals are power signals

Example
Determine whether or not each of the following signal is periodic. If a signal is periodic, specify its fundamental period. i) $x(n) = e^{j6\pi n}$ ii) $x(n) = e^{j3/5(n+1/2)}$ iii) $x(n) = \cos\left(\frac{2\pi}{3}\right)n$

Sol

$\omega_0 = 2\pi f_0$. $f_0 = \frac{6\pi}{2\pi} = 3$.

i) $x(n) = e^{j6\pi n}$

$\omega_0 = 6\pi$. The fundamental freq. is multiple of $\pi$. ∴ the signal is periodic.

$$f_0 = \frac{k}{N} \Rightarrow N = \frac{k}{f_0} = \frac{k}{3}$$

→ (integer)

→ fundamental period (integer)

The minum value of $k$ for which $N$ is integer is 3.

∴ The fundamental period is 1

ii) $x(n) = e^{j3/5(n+1/2)}$

$\omega_0 = 2\pi f_0 = 3/5$ which is not a multiple of $\pi$. ∴ the signal is A periodic.

iii) $x(n) = \cos\left(\frac{2\pi}{3}n\right)$

$\omega_0 = 2\pi f_0 = \frac{2\pi}{3}$ ; The signal is periodic

The fundamental period

$$N = \frac{k}{f_0} = 3k$$

for $k = 1$. $N = 3$. $\therefore$ the fundamental period of the signal is 3.

iv) $x(n) = \cos\left(\frac{\pi}{3}n\right) + \cos\left(\frac{3\pi}{4}\right)n$ $\quad \omega_0 = 2\pi f_0 = \frac{1}{3}\pi$

The fundamental period of the signal $\cos\left(\frac{\pi}{3}n\right)$.

$f_0 = \frac{1}{6}$.

$$N_1 = \frac{k}{f_0} = 6k$$

for $k = 1$ $\quad N_1 = 6$.

The fundamental period of the signal $\cos\left(\frac{3\pi}{4}n\right)$.

$\omega_0 = 2\pi f_0 = \frac{3}{4}\pi$

$f_0 = \frac{3}{8}$

$$N_2 = \frac{k}{f_0} = \frac{8k}{3}.$$

for $k = 3$ $\quad N_2 = 8$.

$$\frac{N_1}{N_2} = \frac{6}{8} = \frac{3}{4} \Rightarrow N = 4N_1 = 3N_2$$
$$= 4 \times 6 = 3 \times 8$$
$$\therefore N = 24$$

Determine the fundamental period of the following signal if they are periodic.

i) $x(n) = \sin\left(\frac{\pi n}{4}\right)$ ii) $x(n) = e^{j2n}$ iii) $x(n) = \cos\left(\frac{\pi}{4}n\right) + \cos 2n$

Causal & Noncausal signals

A signal $x(n)$ is said to be causal if its value is zero for $n < 0$. Otherwise the signal is non causal.

Eg.

For causal

$$x_1(n) = a^n u(n)$$

$$x_2(n) = \{\underset{\uparrow}{1}, 2, -3, -1, 2\}$$

For Non causal

$$x_1(n) = a^n u(-n+1)$$

$$x_2(n) = \{1, -2, \underset{\uparrow}{1}, 4, 3\}$$

A signal that is zero for all $n \geq 0$ is called Anti causal signal.

Simple manipulations of discrete time signals.

Manipulations involve the independent variable (n) & the signal amplitude.

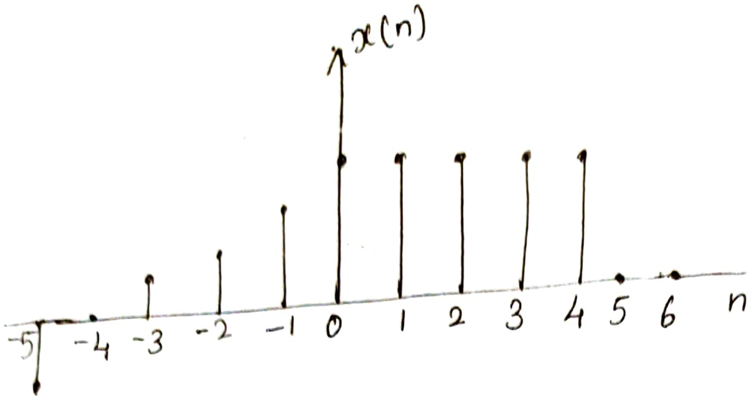
1 Transformation of the independent variable: (Shifting in time)

A signal $x(n)$ may be shifted in time by replacing the independent variable $n$ by $n-k$, where $k$ is an integer.

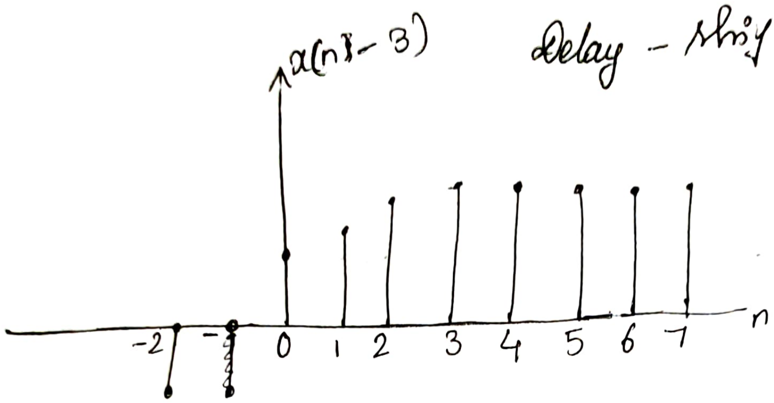
If $k$ is a positive integer, the time shift results in a delay of the signal by $k$ units of time.

If $k$ is a negative integer, the time shift results in an advance of the signal by $|k|$ units in time.

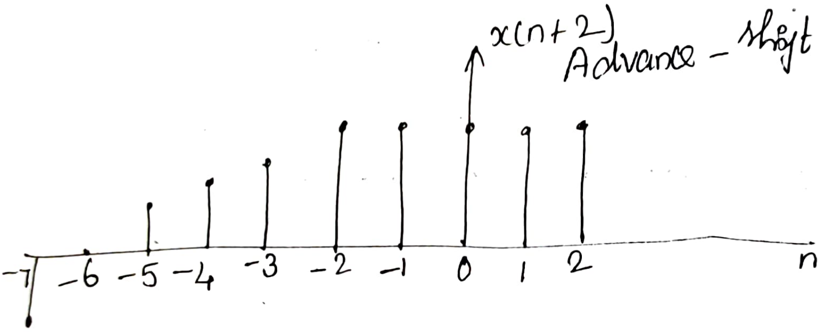
Eg:




Delay - shift to right




Advance - shift to left.




If the signal x(n) is stored on magnetic tape or on a disk or perhaps in the memory of a computer, it is a relatively simple operation to modify the base by introducing a delay or an advance.

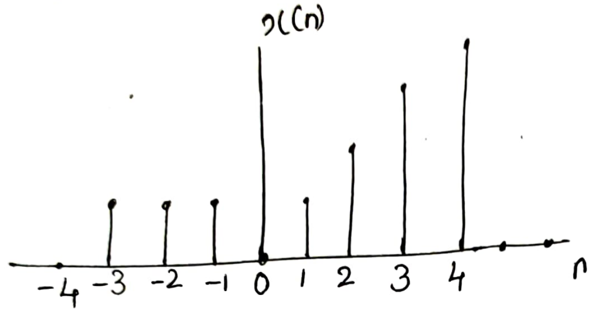
Suppose if the signal is being generated by some physical phenomenon in real time, it is not possible to advance the signal in time, whereas it is always possible to insert a delay into signal samples have already been generated, it is physically impossible to view the future signal sample.

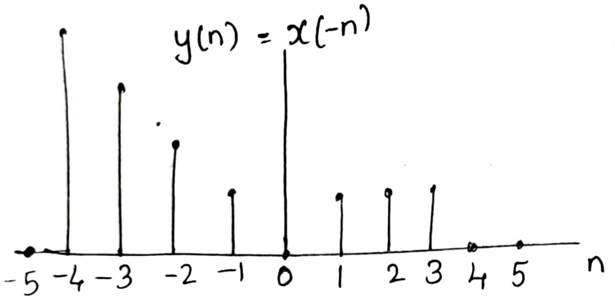
Advancing the real time signal time base is physically unrealizable

2. Folding (or) Reflection

The independent variable $n$ of a discrete time signal $x(n)$ replaced by $-n$ about the time origin is called folding. The result of this operation is a folding or a reflection of the signal about the time origin $n=0$.

Eg: Graphical representation of signal $x(n)$, $x(-n)$ & $x(-n+$



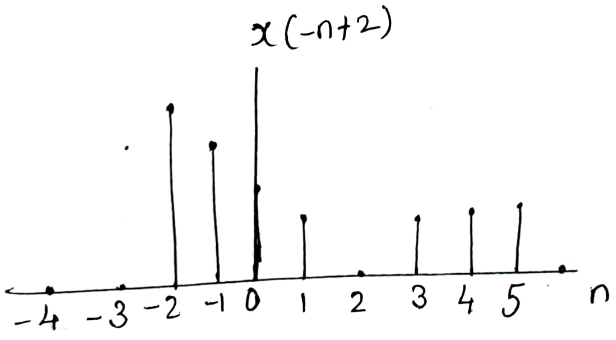

In folding

$x(0) = y(0)$ $\quad$ $x(1) = y(-1)$

$x(-1) = y(1)$ $\quad$ $x(2) = y(-2)$

$x(-2) = y(2)$ $\quad$ $x(3) = y(-3)$

$\vdots$ $\quad$ $\vdots$




Note:

Operations of folding & time delaying a signal are not commutative.

$$TD_k[x(n)] = x(n-k) \quad k>0$$

$$FD[x(n)] = x(-n)$$

$$TD_k\{FD[x(n)]\} = TD_k[x(-n)] = x(-n+k)$$

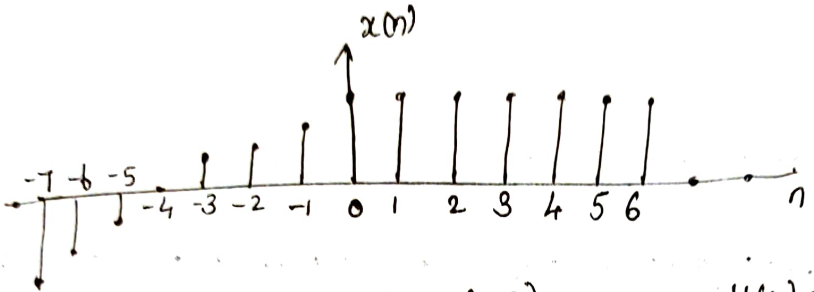
$$FD\{TD_k[x(n)]\} = FD[x(n-k)] = x(-n-k)$$

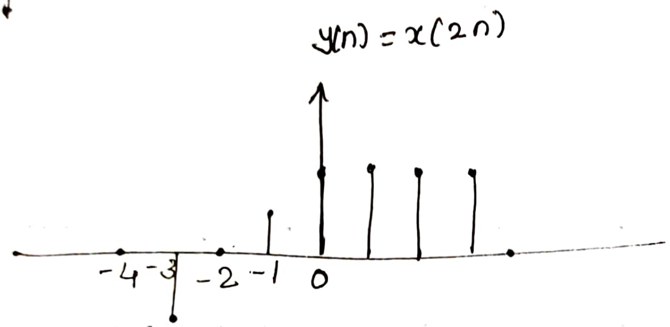
Both results are different.

3. Time scaling (or) Down sampling:

The modification of the independent variable involves replacing $n$ by $\mu n$, where $\mu$ is an integer referred as time scaling or down sampling.

Eg. Graphical representation of the signal $x(n)$ & down sampling operation.




$y(0) = x(0)$ $\quad y(-1) = x(-2)$

$y(1) = x(2)$ $\quad y(-2) = x(-4)$

$y(2) = x(4)$ $\quad y(-3) = x(-6)$

$y(3) = x(6)$ $\quad \vdots$

$y(4) = x(8)$

In this example odd numbered samples in $x(n)$ are skipped & retained the even-numbered samples.

If the signal $x(n)$ was originally obtained by

$x_a(t) \xrightarrow{\text{sampling}} x(n) = x_a(nT)$. where $T$ - sampling interval

Now $y(n) = x(2n) = x_a(2Tn)$ . $\therefore$ The time scaling operation is equivalent to changing the sampling rate from $\frac{1}{T}$ to $\frac{1}{2T}$ ie. decreasing the rate by a factor of 2.

This is a down sampling operation.

4. Addition, Multiplication & Scaling of sequences:

Amplitude modifications include addition, multiplication & scaling of discrete-time signals.

Amplitude scaling:

Amplitude scaling of a signal by a constant A is accomplished by multiplying the value of every signal sample by A.

$$y(n) = A\,x(n) \qquad -\alpha \leq n < \alpha$$

Addition:

The sum of two signals $x_1(n)$ & $x_2(n)$ is a signal $y(n)$, whose value at any instant is equal to the sum of the values of these two signals at that instant

$$y(n) = x_1(n) + x_2(n) \qquad -\alpha < n < \alpha$$

Multiplication.

The product of two signal is defined on a sample to sample basis as

$$y(n) = x_1(n)\,x_2(n) \qquad -\alpha < n < \alpha$$

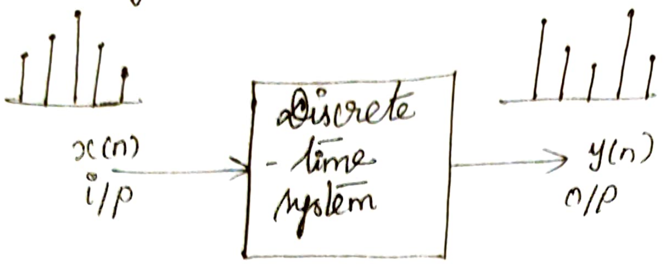
Discrete-Time System:

A device or an algorithm that performs some prescribed operation on a discrete-time signal is called a discrete-time system.

The input signal $x(n)$ is transformed also called an operation by the system into a signal $y(n)$.

$$y(n) = T[x(n)]$$

Block diagram representation of a discrete-time system




Input-Output description of system:

Description of a discrete-time system consist of a mathematical expression or a rule, which explicitly defines the relation between the input & output signals.

The general input-output relationship

$$x(n) \xrightarrow{T} y(n)$$

Determine the response of the following systems to the input signal

$$x(n) = \begin{cases} |n| & -3 \leq n \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

a) $y(n) = x(n)$ b) $y(n) = x(n-1)$ c) $y(n) = x(n+1)$

d) $y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$

e) $y(n) = \max[x(n+1), x(n), x(n-1)]$

f) $y(n) = \sum_{k=-\infty}^{n} x(k) = x(n) + x(n-1) + x(n-2) + \ldots$

Sol

Input signal $x(n) = \{\ldots 0, 3, 2, 1, \underset{\uparrow}{0}, 1, 2, 3, 0, \ldots\}$

a) Output is exactly the 'same' as the input signal. Such a system is known as the identity system.

b) The system simply 'delays' the input by one sample. The output of this system

$$y(n) = \{\ldots 0, 3, 2, \underset{\uparrow}{1}, 0, 1, 2, 3, \ldots\}$$

c) The system 'advances' the input one sample into the future. The response of this system is given by

$$y(n) = \{\dots 0, 3, 2, 1, 0, \underset{\uparrow}{1}, 2, 3, \dots\}$$

d) The output of this system at any time is the mean value of the present, the immediate past & the immediate future samples.

$$y(0) = \frac{1}{3}[x(-1) + x(0) + x(1)]$$

$$= \frac{1}{3}[1+0+1] = \frac{2}{3}$$

similarly $y(1), y(2) \dots$

The output signal is

$$y(n) = \{\dots 0, 1, 5/3, 2, 1, \underset{\uparrow}{2/3}, 1, 2, 5/3, 1, 0, \dots\}$$

e) This system selects as its output at time n the max. value of the three input samples $x(n-1)$, $x(n)$ & $x(n+1)$. Thus the response of this system is

$$y(n) = \{0, 3, 3, 3, 2, 1, 2, 3, 3, 3, 0 \dots\}$$

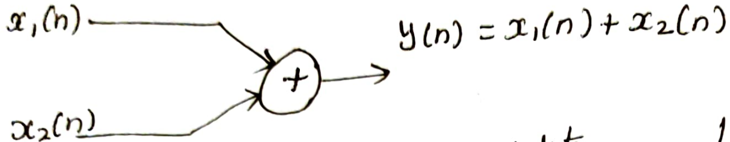
f) This system is basically accumulator that computes the running sum of all the past input values upto present time. The response of the system is given by

$$y(n) = \{\dots 0, 3, 5, 6, 6, 7, 9, 12, 0, \dots\}$$

# Block diagram representation of discrete time systems

Block diagram representation of basic building blocks of discrete-time systems can be interconnected to form complex systems.

## An Adder:



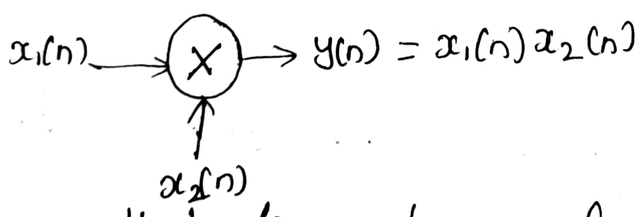

- Adder performs the addition of two signal seq. to form another sequence which is denoted by $y(n)$.
- It is not necessary to store either one of the seq in order to perform addition. i.e. Addition operation is "memoryless".

## A Constant Multiplier:

$$x(n) \xrightarrow{a} y(n) = a x(n)$$

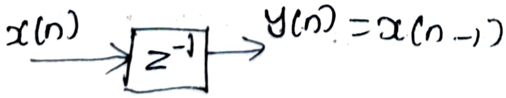
- Applying a scale factor on the input $x(n)$.
- Is a memoryless operation.

## A signal Multiplier:




- Multiplication of two signal sequences to form another sequence called the product.
- Memoryless operation.

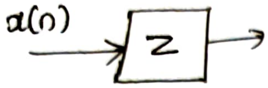
## A unit delay element:




where $z^{-1}$ - unit delay

- Special system, the simply delays the signal pa through it by one sample.
- The sample $x(n-1)$ is stored in memory at time r & it is recalled from memory at time n to for Thus it requires memory.

**A Unit advance element:**




z - Unit advance

- A unit advance moves the input $x(n)$ ahead by one sample in time to yield $x(n+1)$.
- Advance is physically impossible in real time as it involves looking into the future of the signal.
- If the signal is stored in the memory of the computer, any sample at any time can be recallec
- Non real time applications, it is possible to advar the signal $x(n)$ in time.

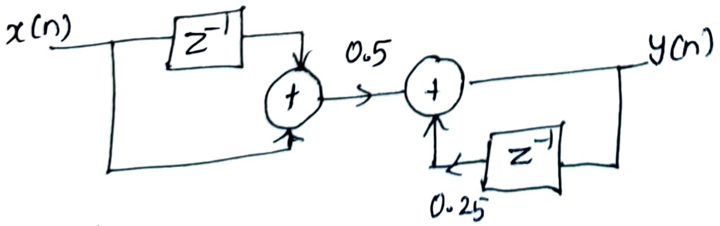
Eg:
Sketch the block diagram representation of the discrete time system described by the i/p - o/p relatio

$$y(n) = \frac{1}{4} y(n-1) + \frac{1}{2} x(n) + \frac{1}{2} x(n-1).$$

where $x(n)$ - i/p & $y(n)$ - o/p of the system

Sol

$$y(n) = 0.25y(n-1) + 0.5x(n) + 0.5x(n-1)$$



Classification of discrete-time systems:

- In the analysis as well as in the design of systems, it is desirable to classify the systems according to the general properties that they satisfy.

- In fact, the mathematical tech. for analyzing & designing discrete-time systems heavily depend on the general characteristics of the systems that are being considered.

Some of the classifications of discrete time systems

1. Static versus dynamic systems:

A discrete-time system is called 'static' or memoryless if its output at any instant n depends at most on the input sample at the same time, but not on past or future samples of the input. In any other case, the system is said to be dynamic or to have memory.

Eg.

Static Systems:

$y(n) = a\,x(n)$

$y(n) = n\,x(n) + b\,x^3(n)$

Both are static or memoryless. Note that there is no need to store any of the past inputs or outputs in order to compute the recent output.

Dynamic Systems:

$y(n) = x(n) + 3x(n-1)$ } - finite memory

$y(n) = \sum_{k=0}^{n} x(n-k)$

$y(n) = \sum_{k=0}^{\infty} x(n-k)$ - infinite memory.

The above systems are dynamic systems or systems of infinite memory

- If the output of a system at time n is completely determined by the input samples in the interval from n-N to N $(N \geq 0)$, the system is said to have memory of duration N. If $N=0$, the system is static.
- If $0 < N < \infty$ the system is said to have finite memory whereas if $N = \infty$ the system is said to have infinite memory.

2. Time - invariant versus Time - variant systems:

A system is called time - invariant if its input - output characteristics do not change with time.

To elaborate

Consider a system T in a relaxed state which when excited by an input signal $x(n)$, produces an output signal

$$y(n) = T[x(n)]$$

Suppose, the same input signal is delayed by k units of time to yield $x(n-k)$ & again applied to the same system. If the characteristics of the system do not change with time, the output of the relaxed system will be

$$y(n-k) = T[x(n-k)]$$

ie The output will be the same as the response to $x(n)$, except that it will be delayed by the same k units in time that the input was delayed.

This leads to define a time - invariant or shift - invariant system as follows

Theorem: A relaxed system $T$ is time-invariant or shift invariant if & only if $x(n) \xrightarrow{T} y(n)$ implies that $x(n-k) \xrightarrow{T} y(n-k)$ for every input signal $x(n)$ & every time shift $k$.

To determine, the given system is time-invariant.

1. Excite the system with an arbitrary input seq. $x(n)$, which produces an output $y(n)$.
2. Delay the input sequence by same amount $k$ & recompute the output $[y(n,k)] = T[x(n-k)]$
3. Delay the output sequence $y(n)$ by $k$ units $[y(n-k$
4. If the output $y(n,k) = y(n-k)$ for all possible values of $k$, the system is time invariant. On the otherhand, if the output $y(n,k) \neq y(n-k)$ even for one value of $k$, the system is time variant.

Diagrammatic explanation of time invariant

$$x(n) \rightarrow \boxed{T} \xrightarrow{y(n)} \boxed{z^{-k}} \rightarrow y(n-k)$$

$$x(n) \rightarrow \boxed{z^{-k}} \xrightarrow{x(n-k)} \boxed{T} \rightarrow y(n,k).$$

1. Determine the following systems are time-invariant or time variant.

a) $y(n) = x(n) - x(n-1)$

b) $y(n) = n\,x(n)$

c) $y(n) = x(-n)$

d) $y(n) = x(n) \cos \omega_0 n$

Sol

a) The system is described by the input-output equation

$$y(n) = T[x(n)]$$
$$= x(n) - x(n-1) \quad \text{— ①}$$

The delay of $y(n)$ by $k$ units in time

$$y(n-k) = x(n-k) - x(n-k-1). \quad \text{— ②}$$

The response of the system to the delay of the input by $k$ units in time is given by

$$y(n,k) = T[x(n-k)]$$
$$= x(n-k) - x(n-k-1) \quad \text{— ③}$$

Since the right hand sides of eq ② & ③ are identical ie $y(n-k) = y(n,k)$. The system is time-invariant.

b) The input-output equation of the system is

$$y(n) = n\,x(n)$$

Delay of the signal $y(n)$ by $k$ units

$$y(n-k) = (n-k)\,x(n-k). \quad \text{— ①}$$

Response of the system to delayed input $x(n-k)$

$$y(n,k) = n\,x(n-k) \quad \text{— ②}$$

$y(n-k) \neq y(n,k)$ ∴ Time variant.

c) $y(n) = x(-n)$.

$$x(n) \rightarrow \boxed{T} \xrightarrow{y(n)=x(-n)} \boxed{z^{-k}} \rightarrow x(-n-k)$$

output $y(n)$ delayed by $k$ units $\quad y(n-k) = x(-n-k) \quad$ — ①

Response of the system to input $x(n-k)$

$$y(n,k) = x(-(n-k)) = x(-n+k) \quad \text{— ②}$$

$y(n-k) \neq y(n,k)$ ∴ Time Variant.

d) $y(n) = x(n)\cos\omega_0 n$

$y(n)$ delayed by $k$ units

$y(n-k) = x(n-k)\cos\omega_0(n-k)$

$$x(n) \rightarrow \boxed{T} \xrightarrow{y(n)=x(n)\cos\omega_0 n} \boxed{z^{-k}} \rightarrow x(n-k)\cos\omega_0 n$$

Response of the system to delayed input $x(n-k)$

$y(n,k) = x(n-k)\cos\omega_0 n$

$y(n-k) \neq y(n,k)$ Time valiant

2. Test the following systems for time in valiant.

a) $y(n) = x(n) + c$ T.I

b) $y(n) = n\,x^2(n)$ T.V

c) $y(n) = a^{x(n)}$ T.I

d) $y(n) = \sum_{k=0}^{m} a(k)x(n-k) - \sum_{k=1}^{m} b(k)y(n-k)$ T.I.

## Linear versus Nonlinear systems:

A linear system is one that satisfies the superposition principle.

**Principle of Superposition:**

The response of the system to a weighted sum of signals be equal to the corresponding weighted sum of the responses of the systems to each of the individual input signal.

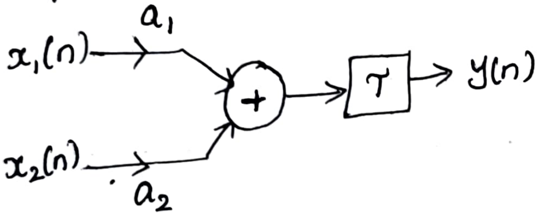
**Theorem:**

A system is linear if & only if

$$T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)]$$

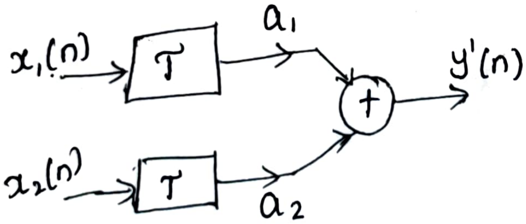
for any arbitary input sequences $x_1(n)$ & $x_2(n)$ & any arbitary constants $a_1$ & $a_2$.

Graphical representation of the superposition principle.




T is linear if & only if

$y(n) = y'(n)$




If a relaxed system does not satisfy the superposition principle as given by the definition, it is called 'Nonlinear'.

Determine if the systems described by the following input-output equations are linear or non linear

a) $y(n) = nx(n)$ d) $y(n) = Ax(n) + B$

b) $y(n) = x(n^2)$ e) $y(n) = e^{x(n)}$

c) $y(n) = x^2(n)$

sol

a) The system T is represented by the input-output equation

$$y(n) = nx(n)$$

$$y(n) = T[x(n)] = nx(n)$$

Consider two input signal $x_1(n)$ & $x_2(n)$. The corresponding outputs are

$$y_1(n) = nx_1(n)$$
$$y_2(n) = nx_2(n)$$

Linear combination of two o/p's results in

$$a_1y_1(n) + a_2y_2(n) = a_1nx_1(n) + a_2nx_2(n).$$

Linear combination of the input sequence. results in the output

$$y_3(n) = T[a_1x_1(n) + a_2x_2(n)]$$
$$= a_1nx_1(n) + a_2nx_2(n)$$

Both the results are idential, the system is linear.

b) The input-output description of the system T is

$$y(n) = T[x(n)] = x(n^2)$$

The response of the system to two arbitrary signals $x_1(n)$ & $x_2(n)$ are

$$y_1(n) = x_1(n^2)$$
$$y_2(n) = x_2(n^2)$$

Linear combination of the two o/p's yields

$$a_1 y_1(n) + a_2 y_2(n) = a_1 x_1(n^2) + a_2 x_2(n^2) \quad - ①$$

The o/p of the system to a linear combination of $x_1(n)$ & $x_2(n)$ is

$$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)]$$
$$= a_1 x_1(n^2) + a_2 x_2(n^2) \quad - ②$$

equation ① & ② are same. ∴ The system is linear.

c) The response of the system

$$y(n) = T[x(n)] = x^2(n)$$

The response of the system to two arbitrary signals $x_1(n)$ & $x_2(n)$ are

$$y_1(n) = x_1^2(n) ; y_2(n) = x_2^2(n)$$

A linear combination of outputs results in the output

$$a_1 y_1(n) + a_2 y_2(n) = a_1 x_1^2(n) + a_2 x_2^2(n) \quad - ①$$

The response of the system to linear combination of the input

$$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)]$$
$$= a_1^2 x_1^2(n) + a_2^2 x_2^2(n) \quad - ②$$

eq ① & ② are not identical. ∴ The system is Nonlinear.

d) The response of the system is

$$y(n) = Ax(n) + B$$

The response of the system to two separate i/p signals $x_1(n)$ & $x_2(n)$ are

$$y_1(n) = Ax_1(n) + B$$

$$y_2(n) = Ax_2(n) + B$$

The linear combination of the o/p results is

$$a_1 y_1(n) + a_2 y_2(n) = a_1 A x_1(n) + a_1 B + a_2 A x_2(n) + a_2 B \quad \text{— ①}$$

The response of the system to linear combination of the i/p is

$$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)]$$

$$= A[a_1 x_1(n) + a_2 x_2(n)] + B \quad \text{— ②}$$

eq ① ≠ ② ∴ The system is Nonlinear.

e) The system is described by the i/p - o/p equation

$$y(n) = e^{x(n)}$$

The response of the system to the i/p's $x_1(n)$ & $x_2(n)$ are $y_1(n) = e^{x_1(n)}$ ; $y_2(n) = e^{x_2(n)}$

The linear combination of o/p's results is

$$a_1 y_1(n) + a_2 y_2(n) = a_1 e^{x_1(n)} + a_2 e^{x_2(n)} \quad \text{— ①}$$

The response of the system to the linear combination of i/p's.

$$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)]$$

$$= e^{a_1 x_1(n) + a_2 x_2(n)} \quad \text{— ②}$$

eq ① ≠ ②. ∴ The system is Nonlinear.

Causal & Noncausal systems:

A system is said to be causal if the output of the system at any time n depends only on present & past inputs [$x(n)$, $x(n-1)$, $x(n-2)$, ...] but does not depend on future inputs [$x(n+1)$, $x(n+2)$, ...].

In mathematical terms, the output of a causal system satisfies an equation of the form

$$y(n) = F[x(n), x(n-1), x(n-2), \ldots]$$

where $F[\cdot]$ is some arbitrary function.

If a system does not satisfy this definition, it is called noncausal. Such a system has an output that depends not only on present & past but also on future inputs.

Note:

In real-time signal processing applications it is not possible to observe future values of the signal & hence a noncausal system is physically unrealizable ie can't be implemented.

If the signal is recorded so that the processing is done off-line, it is possible to implement a noncausal often the case is the processing of geophysical signals & images.

Eg:

Determine if the systems described by the following input-output equations are causal or noncausal.

a) $y(n) = x(n) - x(n-1)$ d) $y(n) = x(n) + 3x(n+4)$

b) $y(n) = \sum_{k=-\infty}^{n} x(k)$ e) $y(n) = x(n^2)$

f) $y(n) = x(2n)$

c) $y(n) = a x(n)$ g) $y(n) = x(-n)$

Sol:

a) $y(n) = x(n) - x(n-1)$

b) $y(n) = \sum_{k=-\infty}^{n} x(k)$ } - Causal systems.

c) $y(n) = a x(n)$

d) $y(n) = x(n) + 3x(n+4)$

e) $y(n) = x(n^2)$ } - Noncausal systems

f) $y(n) = x(2n)$

g) $y(n) = x(-n)$ - The output at $n=-1$ depends on the input at $n=1$, which is two units of time into the future. ∴ Noncausal.

Stable & Unstable system:

Stability is an important property that must be considered in any practical application of a system.

Unstable systems usually exhibit erratic & extreme behavior & cause overflow in any practical implementation.

Def:

An arbitrary relaxed system is said to be bounded input - bounded output (BIBO) stable if & only if every bounded input produces a bounded output.

The condition that the input sequence $x(n)$ & the output sequence $y(n)$ are bounded is translated mathematically to mean that there exist some finite numbers say $M_x$ & $M_y$ such that

$$|x(n)| \leq M_x < \infty \; ; \; |y| \leq M_y < \infty \text{ for all } n.$$

If for some bounded input sequence $x(n)$, the output is unbounded, the system is classified as unstable.

By convolution sum formula

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

Taking absolute values on both sides of equation, we get

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \right|$$

$$= \sum_{k=-\infty}^{\infty} |h(k)|\, |x(n-k)|$$

If the input is bounded then $|x(n-k)| = M_x$

$$\therefore |y(n)| = \sum_{k=-\infty}^{\infty} |h(k)|\, M_x$$

$$= M_x \sum_{k=-\infty}^{\infty} |h(k)|$$

Test the stability of the following systems

a) $y(n) = \cos[x(n)]$ b) $y(n) = x(-n-2)$ c) $y(n) = n x(n)$

Sol:

a) Given that $y(n) = \cos[x(n)]$

For stability of a system the impulse response shd be absolutely summable ie $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

The impulse response is obtained when the input is unit impulse signal $\delta(n)$ ie if $x(n) = \delta(n)$ then $y(n) = h(n)$

$\therefore$ Impulse response $h(n) = \cos[\delta(n)]$

when $n = 0$ ; $h(0) = \cos(1) = 0.5403$

$n = 1$ ; $h(1) = \cos 0 = 1$

$n = 2$ ; $h(2) = \cos 0 = 1$

$\vdots$

$n = \infty$ ; $h(\infty) = \cos 0 = 1$

$n = -1$ ; $h(-1) = \cos 0 = 1$

$\vdots$

$n = -\infty$ ; $h(-\infty) = \cos 0 = 1$

$\therefore \sum_{k=-\infty}^{\infty} |h(k)| = h(-\infty) \ldots + h(-1) + h(0) + h(1) + h(2) + \ldots h(\infty)$

$= 1 + \ldots + 1 + 1 + 0.5403 + 1 + 1 + \ldots + 1$

$= \infty$

Since $\sum_{k=-\infty}^{\infty} |h(k)|$ tends to infinity the system is unstable.

b) Impulse response $h(n) = \delta(-n-2)$

$\therefore$ when $n = 0$ ; $h(0) = \delta(-2) = 0$

$n = 1$ ; $h(1) = \delta(-3) = 0$

$\vdots$

$n = \infty$ ; $h(\infty) = \delta(-\infty) = 0$

$n = -1$ ; $h(-1) = \delta(-1) = 0$

$n = -2$ ; $h(-2) = \delta(0) = 1$

$n = -\infty$ ; $h(-\infty) = \delta(\infty) = 0$

$$\therefore \sum_{k=-\infty}^{\infty} |h(k)| = 0+0+0+\ldots+1+0+0\ldots+0$$
$$= 1$$

$\sum_{k=-\infty}^{\infty} |h(k)|$ is less than infinity. Hence the system is stable.

c) Impulse Response $h(n) = n\delta(n)$

$\therefore$ when $n=0$; $h(0) = 0\delta(0) = 0$

$n=1$; $h(1) = 1\delta(1) = 0$

$\vdots$

$n=\infty$; $h(\infty) = \infty\delta(\infty) = 0$

$n=-1$; $h(-1) = (-1)\delta(-1) = 0$

$\vdots$

$n=-\infty$; $h(-\infty) = (-\infty)\delta(-\infty) = 0$

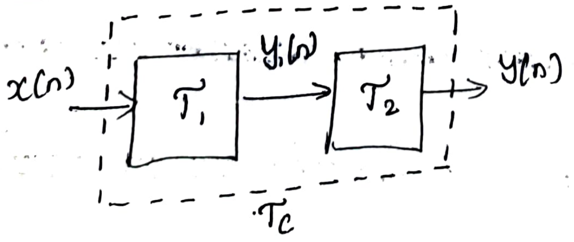
$$\sum_{k=-\infty}^{\infty} |h(k)| = \ldots + h(-2) + h(-1) + h(0) + h(1) + \ldots$$
$$= 0$$

$\sum_{k=-\infty}^{\infty} |h(k)|$ is less than infinity. Hence the system is stable.

Interconnection of discrete-time systems

- Interconnected to form large systems.
- Two basic ways to interconnect.
  1. Cascade 2. Parallel.

Cascade:



Output of the first system is

$$y_1(n) = T_1[x(n)]$$

Output of the second system is

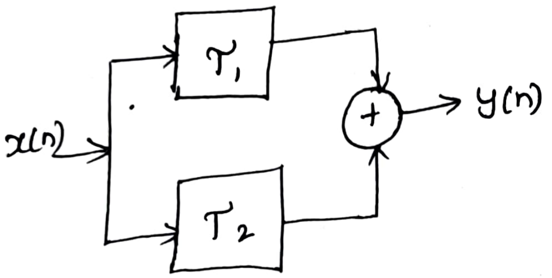
$$y(n) = T_2[y_1(n)]$$

$$= T_2[T_1[x(n)]] = T_c[x(n)]$$

Also $T_2 T_1 \neq T_1 T_2$ for any arbitrary system

If $T_1$ & $T_2$ - linear time invariant then $T_c$ also time invariant & $T_2 T_1 = T_1 T_2$.

Parallel:




$$y(n) = y_1(n) + y_2(n)$$

$$= T_1[x(n)] + T_2[x(n)]$$

$$= (T_1 + T_2)[x(n)]$$

$$= T_p[x(n)]$$

## Linear Time invariant systems:

System satisfies the linearity & time-invariant property.

Techniques for the analysis of linear system.

Two basic methods for analyzing the behavior or response of a linear system.

### First Method

<u>Direct solution</u> of the input-output eq. for the system.

The general form of input-output equation of a system is

$$y(n) = F\left[y(n-1), y(n-2), \dots y(n-N), x(n), x(n-1) \dots x(n-M)\right]$$

where $F[\cdot]$ - some arbitrary function

Specifically, for an LTI system

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

where $\{a_k\}$ & $\{b_k\}$ - constant parameters that specify the system & are independent of $x(n)$ & $y(n)$.

This i/p-o/p relationship is called a difference equation & represents one way to characterize the behaviour of a discrete-time LTI system

Second Method:

Discrete convolution for analyzing the behaviour of a linear system to a given i/p signal.

1. The i/p signal $x(n)$ is resolved into a weighted sum of elementary signal components $\{x_k(n)\}$ so that

$$x(n) = \sum_k c_k x_k(n)$$

where $\{c_k\}$ - set of amplitudes in the decomposition of the signal $x(n)$.

2. The response of the system to the elementary signal component $x_k(n)$ is $y_k(n)$. Thus

$$y_k(n) = T[x_k(n)]$$

Assume that the system is relaxed so that the response to $c_k x_k(n)$ is $c_k y_k(n)$ as a consequence of the scaling property of the linear system.

3. The total response to the input $x(n)$ is

$$y(n) = T[x(n)]$$

$$= T\left[\sum_k c_k x_k(n)\right]$$

$$= \sum_k c_k T[x_k(n)]$$

↳ Using the additivity property of the linear system.

$$= \sum_k c_k y_k(n).$$

Eg:
Consider the finite-duration sequence as $x(n) = \{2, \underset{\uparrow}{4}, 0, 3\}$. Resolve the sequence $x(n)$ into a sum of weighted impulse sequence.

Sol:
The nonzero time instants $n = -1, 0, 2$.

$\therefore$ Three impulses are needed at delay $k = -1, 0, 2$

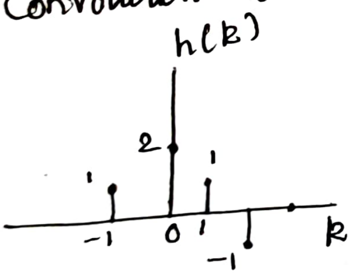
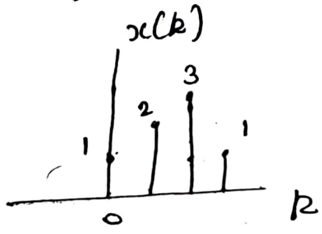
$$x(n) = 2\delta(n+1) + 4\delta(n) + 3\delta(n-2)$$

The impulse response of a linear time-invariant system is $h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$. Determine the response of the system to the input signal
$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$

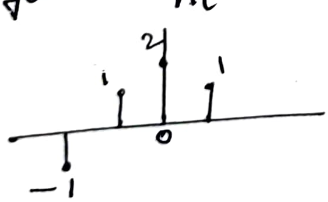
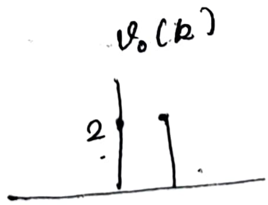
Sol:
The response of the system is given by

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

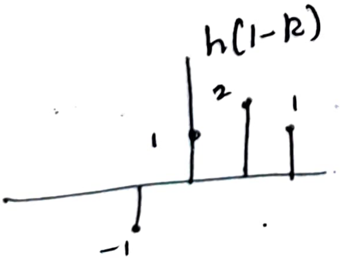
Convolution is computed using the graphical method

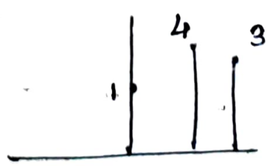
fold ⇓ Multiply

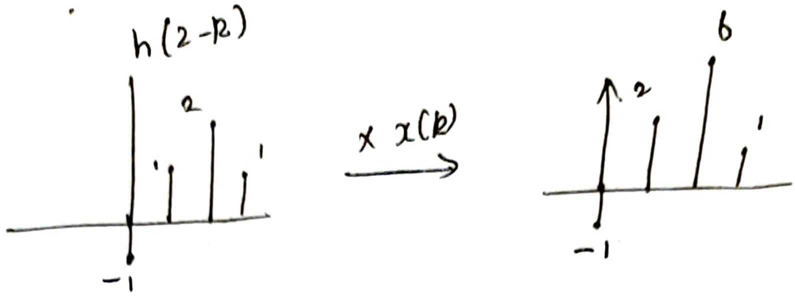
$y(0) = 4$



$\times\, x(k)$



$y(1) = 8$




$y(2) = 8$

$y(n) = \{ \ldots \; 0, 0, 1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1, 0, 0, \ldots \}$

1. Determine the output $y(n)$ of a relaxed linear time invariant system with impulse response $h(n) = a^n u(n)$ $|a| < 1$. when the input step sequence is $x(n) = u(n)$

2. $x(n) = \{\dot{3}, 2, 1, 2\}$ $h(n) = \{1, 2, 1, 2\}$

<u>sol</u> $y(n) = \{3, \underset{\uparrow}{8}, 8, 12, 9, 4, 4\}$

3. Find the convolution of the signals

$x(n) = 1 \quad n = -2, 0, 1$

$= 2 \quad n = -1$

$= 0 \quad$ otherwise

$h(n) = \delta(n) - \delta(n-1) + \delta(n-2) - \delta(n-3)$

<u>sol</u> $x(n) = \{1, 2, \underset{\uparrow}{1}, 1\}$

$h(n) = \{\underset{\uparrow}{1}, -1, 1, -1\}$

$y(n) = \{1, 1, \underset{\uparrow}{0}, 1, -2, 0, -1\}$

Third Method:

Analysis of discrete time system using z-transform.

The convolution property of z transform says that the z-transform of the convolution of $x(n)$ & $h(n)$ is equal to the product of their individual z transform.

$$Z\{x(n) * h(n)\} = X(z)H(z)$$

since $y(n) = x(n) * h(n)$

$$Z\{y(n)\} = Z\{x(n) * h(n)\}$$

$$\therefore Y(z) = X(z)H(z). \Rightarrow H(z) = \frac{Y(z)}{X(z)}.$$

The response $y(n)$ of LTI system is obtained by taking inverse z transform.

$$y(n) = Z^{-1}\{X(z)H(z)\}$$

$\frac{Y(z)}{X(z)}$ is the transfer function of LTI system which is nothing but z transform of impulse response of the system.

Eg:

1. Determine the impulse response $h(n)$ for the system described by the second order difference equation $y(n) - 4y(n-1) + 4y(n-2) = x(n-1)$.

Sol

The difference equation governing the system is

$$y(n) - 4y(n-1) + 4y(n-2) = x(n-1)$$

taking z transform on the difference eq

$$Y(z) - 4z^{-1}Y(z) + 4z^{-2}Y(z) = z^{-1}X(z).$$

$$Y(z)\left(1-4z^{-1}+4z^{-2}\right) = z^{-1}X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{z^{-1}}{1-4z^{-1}+4z^{-2}}$$

$$H(z) = \frac{z^{-1}}{z^{-2}\left(z^{2}-4z+4\right)}$$

$-2z^2+4$

$$= \frac{z}{z^{2}-4z+4}$$

$(z-2)^2$

$$= \frac{z}{(z-2)^{2}}$$

$$= \frac{1}{2}\,\frac{2z}{(z-2)^{2}} \qquad \because Z\left\{na^{n}\right\} = \frac{az}{(z-a)^{2}}$$

By taking inverse Z transform

The impulse response $h(n) = Z^{-1}\left\{H(z)\right\}$

$$= Z^{-1}\left\{\frac{1}{2}\,\frac{2z}{(z-2)^{2}}\right\}$$

$$= \frac{1}{2}\, n\, 2^{n} u(n)$$

$$= 2^{-1}\, n\, 2^{n} u(n)$$

$$h(n) = n\,2^{n-1} u(n) \text{ for all } n$$

2. Find the transfer function & unit sample response of the second order difference with zero initial condition $y(nT) = x(nT) - 0.25y(nT-2T)$

Sol:

The difference equation governing the system is

$$y(nT) = x(nT) - 0.25y(nT-2T)$$

On taking z transform

$$Y(z) = X(z) - 0.25z^{-2}Y(z)$$

$$Y(z) + 0.25z^{-2}Y(z) = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1+0.25z^{-2}}$$

$$= \frac{1}{z^{-2}(z^2+0.25)}$$

$$= \frac{z^2}{(z+j0.5)(z-j0.5)}$$

$$\frac{H(z)}{z} = \frac{z}{(z+j0.5)(z-j0.5)}$$

By taking partial fraction

$$\frac{H(z)}{z} = \frac{0.5}{z+j0.5} + \frac{0.5}{z-j0.5}$$

$$H(z) = 0.5\left[\frac{z}{z+j0.5} + \frac{z}{z-j0.5}\right]$$

taking inverse z transform

Impulse Response $h(n) = Z^{-1}\{H(z)\}$

$$= 0.5\left[(-j0.5)^n u(n) + (j0.5)^n u(n)\right]$$

for all values of n..

3. Determine the impulse response of the discrete time LTI system defined by

$$y(nT) - 2y(nT-T) + y(nT-2T) = x(nT) + 3\delta(nT-3T)$$

## Discrete Time fourier transform :

The fourier transform of a finite energy discrete time signal $x(n)$ is defined as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad -①$$

where $x(n)$ - discrete time signal

$X(\omega)$ (or) $X(e^{j\omega})$ - Fourier transform of $x(n)$

Physically

- $X(\omega)$ represents the frequency content of the signal $x(n)$ (or) $X(\omega)$ is a decomposition of $x(n)$ into its freq components.
- Since the computation of FT involves summing infinite number of terms, the FT exists only for the signals that are absolutely summable. ie for given $x(n)$, the $X(\omega)$ exists only when $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$

Two basic difference b/w the FT of discrete time & analog signals. (finite energy)

| Continuous time signal | Discrete time signal |
|---|---|
| 1. Consist of a spectrum with a freq. range $-\infty$ to $+\infty$ | Unique in the freq. range $-\pi$ to $+\pi$ or 0 to $2\pi$. ∴ Periodic. |
| 2. Since continuous, FT involves integration | Involve summation becoz the signals are discrete. |

Inverse fourier transform:

Evaluate the sequence $x(n)$ from $X(\omega)$. Multiply both sides of $X(\omega)$ by $e^{j\omega m}$ & integrate over the interval $-\pi$ to $\pi$. Thus

$$\int_{-\pi}^{\pi} X(\omega) e^{j\omega m} d\omega = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} e^{j\omega m} d\omega \quad \text{— (2)}$$

The integral on the right hand side of the equation can be evaluated if we interchange the order of summation & integration. This interchange can be made if the series

$$X_N(\omega) = \sum_{n=-N}^{N} x(n) e^{-j\omega n}$$

convergerges uniformly to $X(\omega)$ as $N \to \infty$. Uniform convergence means that for every $\omega$ $X_N(\omega) \to X(\omega)$ as $N \to \infty$. By interchanging the order of summation & integration.

$$\int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = \begin{cases} 2\pi & m = n \\ 0 & m \neq n \end{cases}$$

Consequently

$$\sum_{n=-\infty}^{\infty} x(n) \int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = \begin{cases} 2\pi x(m) & m=n \\ 0 & m \neq n \end{cases} \quad (3)$$

Combining eq (2) & (3)

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega.$$

Note:
The FT pair for discre time signal is

$$x(n) \xleftrightarrow{F} X(\omega)$$

Frequency Analysis of discrete-time aperiodic signal

* Analysis Equation (Direct transform)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

* Synthesis Equation (Inverse transform)

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

Fourier transform theorems & properties:

Linearity:

The fourier transform of a linear combination of two or more signals is equal to the same linear combination of the fourier transforms of the individual signals.

This property makes the FT suitable for the study of linear systems.

If $x_1(n) \xleftrightarrow{F} X_1(\omega)$ &

$x_2(n) \xleftrightarrow{F} X_2(\omega)$

then

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{F} a_1 X_1(\omega) + a_2 X_2(\omega)$$

Proof

$$F[a_1 x_1(n) + a_2 x_2(n)] = a_1 F[x_1(n)] + a_2 F[x_2(n)]$$
$$= a_1 X_1(\omega) + a_2 X_2(\omega)$$

**Periodicity:**

The discrete-time fourier transform $X(e^{j\omega})$ is periodic in $\omega$ with period $2\pi$

$$X(e^{j\omega}) = X[e^{j(\omega+2\pi k)}] \text{ for any integer } k.$$

Implication: Only one period of $X(e^{j\omega})$ is needed for analysis ~~is~~ not the whole range $-\infty < \omega < \infty$

**Time Shifting:**

If $x(n) \xleftrightarrow{F} X(\omega)$ then $x(n-k) \xleftrightarrow{F} e^{-j\omega k} X(\omega)$.
where $k$ is an integer.

**Proof**

$$F[x(n-k)] = \sum_{n=-\infty}^{\infty} x(n-k) e^{-j\omega n}$$

Put $n-k = P \Rightarrow n = P+k$.

$$= \sum_{n=-\infty}^{\infty} x(P) e^{-j\omega(P+k)}$$

$$= \sum_{n=-\infty}^{\infty} x(P) e^{-j\omega P} \cdot e^{-j\omega k}$$

$$= e^{-j\omega k} X(\omega).$$

Time shifting of a signal by $k$ units does not change its amplitude spectrum. The phase spectrum is changed by $-\omega k$.

**Frequency Shifting:**

If $F[x(n)] = X(e^{j\omega})$ then

$$F[x(n)e^{j\omega_0 n}] = X[e^{j(\omega-\omega_0)}]$$

Proof

$$F\left[x(n)e^{j\omega_0 n}\right] = \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega_0 n}\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\omega_0)n}$$

$$= X\left[e^{j(\omega-\omega_0)}\right]$$

Time reversal:

If $F[x(n)] = X(e^{j\omega})$ then

$$F[x(-n)] = X(e^{-j\omega})$$

Proof:

$$F[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-(-j\omega)n}$$

$$= X(e^{-j\omega})$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-(-j\omega n)}$$

$$= X(e^{-j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Differentiation in frequency:

If $F[x(n)] = X(e^{j\omega})$ then

$$F[n\,x(n)] = \frac{j\,d}{d\omega} X(e^{j\omega}).$$

Proof:

$$X(e^{j\omega}) = F[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Differentiate both sides with respect to $\omega$.

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} (-jn)\, x(n)\, e^{-j\omega n}$$

$$= -j \sum_{n=-\infty}^{\infty} n\,x(n)\, e^{-j\omega n}$$

$$\frac{1}{-j}\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} n\,x(n)\, e^{-j\omega n}$$

$$j\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} n x(n) e^{-j\omega n}$$

$$= F[n x(n)]$$

Time Convolution:

If $F[x_1(n)] = X_1(e^{j\omega})$ & $F[x_2(n)] = X_2(e^{j\omega})$ then

$$F[x_1(n) * x_2(n)] = \boxed{X_1(e^{j\omega}) X_2(e^{j\omega})}$$

Proof:

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

$$F[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) e^{-j\omega n}$$

By interchanging the order of summation we get,

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k) e^{-j\omega n}$$

Put $n-k = P$ then

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{P=-\infty}^{\infty} x_2(P) e^{-j\omega(P+k)}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{P=-\infty}^{\infty} x_2(P) e^{-j\omega P} \cdot e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) e^{-j\omega k} \sum_{P=-\infty}^{\infty} x_2(P) e^{-j\omega P}$$

$$= X_1(e^{j\omega}) X_2(e^{j\omega})$$

The convolution of two signals in time domain is equal to multiplying their spectra in the frequency domain.

Frequency Convolution:

If $F[x_1(n)] = X_1(e^{j\omega})$ & $F[x_2(n)] = X_2(e^{j\omega})$

then

$$F[x_1(n)x_2(n)] = X_1(e^{j\omega}) \circledast X_2(e^{j\omega})$$

$$= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$

Proof

$$F[x_1(n)x_2(n)] = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_2(n) \left[\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\, e^{j\theta n}\, d\theta\right] e^{-j\omega n}$$

Interchanging the order of summation & integral,

$$= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\left[\sum_{n=-\infty}^{\infty} x_2(n)\, e^{-j(\omega-\theta)n}\right] d\theta$$

$$= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$

This operation is known as periodic convolution because it is the convolution of two periodic fns $X_1(e^{j\omega})$ & $X_2(e^{j\omega})$

The correlation theorem:

If $F[x_1(n)] = X_1(e^{j\omega})$ & $F[x_2(n)] = X_2(e^{j\omega})$ then

$$F[r_{x_1 x_2}(l)] = \Gamma_{x_1 x_2}(e^{j\omega}) = X_1(e^{j\omega})\, X_2(e^{-j\omega})$$ ✓

The function $\Gamma_{x_1 x_2}(e^{j\omega})$ is called the cross energy density spectrum of the signal $x_1(n)$ & $x_2(n)$.

The Modulation theorem:

If $F[x(n)] = X(e^{j\omega})$ then

$$F[x(n)\cos\omega_0 n] = \frac{1}{2}\left[X(e^{j(\omega+\omega_0)}) + X(e^{j(\omega-\omega_0)})\right]$$

Symmetry properties:

The Fourier transform $X(e^{j\omega})$ is a complex function of $\omega$ & can be expressed as

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

where $X_R(e^{j\omega})$ – Real part

$X_I(e^{j\omega})$ – Imaginary part of $X(e^{j\omega})$ respectively.

Parseval's theorem:

If $F[x(n)] = X(e^{j\omega})$ then

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega.$$

## Impulse Response:

The impulse response of a linear time-invariant system was defined as the response of the system to a unit sample excitation [ie $x(n) = \delta(n)$].

The general form of difference equation of a $N^{th}$ order system is given by

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \quad N > M$$

The total response of the system to any excitation function consists of the sum of two solutions of the difference equation: the solution to the <u>homogeneous equation</u> plus the <u>particular solution</u> to the excitation function

The solution $y(n) = y_h(n) + y_p(n)$

In the case where the excitation is an impulse, the particular solution is zero, since $x(n) = 0$ for $n > 0$

$$y_p(n) = 0$$

Consequently, the response of the system to an impulse consists of the solution to the homogeneous equation, with the $\{C_k\}$ parameters evaluated to satisfy the initial conditions dictated by the impulse.

$$y(n) = y_h(n)$$

$h(n)$ is obtained by solving the homogeneous equation

$$\sum_{k=0}^{N} a_k y(n-k) = 0 \; ; \qquad \underline{a_0 = 1}$$

for the case $N > M$. If $N = M$ add an impulse fn to homogeneous solution

Determine the impulse response $h(n)$ for the system described by the second-order difference equation

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1).$$

Sol: $N \geq M$ ; $N = 2$ ; $M = 1$.

Impulse Response $h(n) = y_h(n)$.

To find the homogeneous solution, consider the eq obtained by $x(n) = 0$. Assume $y_h(n) = \lambda^n$

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$$

$$\lambda^{n-2}(\lambda^2 - 3\lambda - 4) = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

The roots are $\lambda = -1, 4$.

The general form of the solution to the homogeneous equation is

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n$$

$$= C_1(-1)^n + C_2(4)^n$$

To evaluate the constants, sub $n = 0$ & $1$.

$$y(0) = C_1 + C_2 \quad \text{①}$$

$$y(1) = C_1(-1) + C_2(4) = -C_1 + 4C_2 \quad \text{②}$$

On the otherhand, from the given difference equation sub $n = 0$.

$$y(0) - 3y(-1) + 4y(-2) = x(0) + 2x(-1)$$

Consider all initial conditions are zero.

$$y(0) = 1 \quad \text{③}$$

$$y(1) - 3y(0) - 4y(-1) = x(1) + 2x(0)$$

$$y(1) = +3 + 2 = +5 \quad \text{④}$$

From eq ①, ②, ③, ...

$$c_1 + c_2 = 1 \quad — ⑤$$
$$-c_1 + 4c_2 = +5 \quad — ⑥$$

⑤ + ⑥

$$\begin{array}{r} \cancel{c_1} + c_2 = 1 \\ -\cancel{c_1} + 4c_2 = +5 \\ \hline 5c_2 = 6 \end{array}$$

$$c_2 = \frac{6}{5}$$

from ⑤

$$c_1 + 6/5 = 1$$
$$c_1 = 1 - 6/5$$
$$c_1 = -1/5$$

$$y(n) = \left[-1/5\,(-1)^n + 6/5\,(4)^n\right] u(n) \;//$$

Determine the impulse response for the system
$$y(n) = 0.9\,y(n-1) + x(n)$$

Order of y poly. $N = 1$
Order of x poly. $M = 0$

Response is given by
$$y(n) = y_h(n) + y_p(n)$$

For impulse response $y_p(n) = 0$
$$y(n) = y_h(n)$$

Since $N > M$ $\quad y_h(n) = c_1\lambda_1^n + c_2\lambda_2^n + \ldots$

make y. poly to zero

$$y(n) - 0.9y(n-1) = 0$$

Let $y_h(n) = y(n) = \lambda^n$

$$\lambda^n - 0.9\lambda^{n-1} = 0$$

$$\lambda^{n-1}(\lambda - 0.9) = 0$$

$$\lambda - 0.9 = 0$$

$$\lambda = 0.9$$

$$\therefore y_h(n) = C_1\lambda^n = C_1(0.9)^n$$

To compute $C_1$ subs. the value of n

$n=0$; $y(0) = C_1(0.9)^0 = C_1$ — ①

Also subs. in difference equation

$$y(n) = 0.9y(n-1) + x(n)$$

$n=0$; $y(0) = 0.9y(-1) + x(0)$ [$x(0) = \delta(0)$ ∵ Impulse Response]

$$= 0.9(0) + 1$$

$$= 1 \quad — ②$$

From eq ① & ②

$$C_1 = 1$$

∴ Impulse Response $y(n) = (0.9)^n u(n)$

Determine the impulse response for the system function

$$y(n) = 0.81y(n-2) + x(n) - x(n-2)$$

Sol:

Order of y poly. $N = 2$

Order of x poly $M = 2$

Response of the system

$$y(n) = y_h(n) + y_p(n)$$

For impulse response $y_p(n) = 0$.

Find the fourier transform of the following.

a) $\left(\frac{1}{2}\right)^{|n-1|}$ b) $\left(\frac{1}{2}\right)^{n-1} u(n-1)$ c) $\delta(n-1)+\delta(n+1)$

d) $\delta(n+2)-\delta(n-2)$

Sol

a) Given $x(n) = \left(\frac{1}{2}\right)^{|n-1|}$

Fourier transform of $\left(\frac{1}{2}\right)^{|n|}$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{|n|} e^{-j\omega n} + \sum_{n=-1}^{-\infty} \left(\frac{1}{2}\right)^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} e^{-j\omega n} + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n} e^{j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-j\omega}\right)^{n} + \sum_{n=1}^{\infty} \left(\frac{1}{2} e^{j\omega}\right)^{n}$$

$$= \left[1 + \frac{1}{2} e^{-j\omega} + \frac{1}{4} e^{-2j\omega} + \dots\right] + \left[\sum_{n=0}^{\infty} \left(\frac{1}{2} e^{j\omega}\right)^{n} - 1\right]$$

$$= \frac{1}{1-0.5e^{-j\omega}} + \left[\frac{1}{1-0.5e^{j\omega}} - 1\right] \quad \left[\because 1 + a + a^2 + \dots \infty = \frac{1}{1-a}\right]$$

$$= \frac{1}{1-0.5e^{-j\omega}} + \frac{\cancel{1} + 0.5e^{j\omega} - \cancel{1}}{1-0.5e^{j\omega}}$$

$$= \frac{1-0.5e^{j\omega} + (1-0.5e^{-j\omega})(+0.5e^{j\omega})}{(1-0.5e^{-j\omega})(1-0.5e^{j\omega})}$$

$$= \frac{1-\cancel{0.5e^{j\omega}} + \cancel{0.5e^{j\omega}} - 0.25}{1-0.5e^{-j\omega} - 0.5e^{j\omega} + 0.25}$$

$$= \frac{0.75}{1.25 - 0.5(e^{-j\omega}+e^{j\omega})} = \frac{0.75}{1.25 - \cos\omega}$$

Using time shifting property

$$F\left[\left(\frac{1}{2}\right)^{|n-1|}\right] = \frac{0.75\, e^{-j\omega}}{\dots}$$

6. Given $x(n) = \left(\frac{1}{2}\right)^{n-1} u(n-1)$

$$F\left[\left(\tfrac{1}{2}\right)^n u(n)\right] = \sum_{n=-\alpha}^{\alpha} \left(\tfrac{1}{2}\right)^n e^{-j\omega n} u(n)$$

$$= \sum_{n=0}^{\alpha} \left(\tfrac{1}{2}\right)^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\alpha} \left(\tfrac{1}{2} e^{-j\omega}\right)^n$$

$$= \frac{1}{1-0.5e^{-j\omega}}$$

Using time shifting property

$$F\left[\left(\tfrac{1}{2}\right)^{n-1} u(n-1)\right] = \frac{e^{-j\omega}}{1-0.5e^{-j\omega}}$$

iii) $F[\delta(n+1) + \delta(n-1)]$

$$= \sum_{n=-\alpha}^{\alpha} \delta(n+1) e^{-j\omega n} + \sum_{n=-\alpha}^{\alpha} \delta(n-1) e^{-j\omega n}$$

$$= e^{j\omega} + e^{-j\omega} = 2\cos\omega.$$

[$\because$ $\delta(n+1) = 1$ for $n = -1$

$\delta(n+1) = 0$ for $n \neq -1$ ]

[$\because$ $\delta(n-1) = 1$ for $n = 1$

$= 0$ for $n \neq 1$.

iv) $F[\delta(n+2) - \delta(n-2)]$

$$= e^{j2\omega} - e^{-2j\omega} = 2j\, s\sin n\omega$$

Find the DTFT of $x(n) = a^n u(n)$.

$$F[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$= \frac{1}{1 - a e^{-j\omega}}$$

# Frequency Response Analysis of Discrete-time sy

The output y(n) of any linear time invariant system to an input signal x(n) can be obtained using convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \quad \text{--- (1)}$$

where h(n) is the impulse response of the system.

Let us consider a complex exponential signal $x(n) = e^{j\omega n}$ as input to the system.

Then the o/p is given by

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)}$$

$$= e^{+j\omega n} \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}$$

$$= e^{j\omega n} \times H(e^{j\omega})$$

where

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}$$

(For an i/p signal $e^{j\omega n}$ the o/p of an LTI-DT system is also exponential signal of the same freq. multiplied by the factor $H(e^{j\omega})$.

These type of signals that produce a response which differ from the i/p signal by a complex constant that are known as eigen function.

The value of the complex constant $H(e^{j\omega})$ evaluated at the freq of the i/p signal is known as eigen values of the system)

- $H(e^{j\omega})$ is also called as freq. response of the system.
- Freq. response of a system is discrete time fourier transform of the impulse response $h(n)$ of the system.
- Since $H(e^{j\omega})$ — Complex valued function, it can be expressed in polar-form as

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}$$

$|H(e^{j\omega})| = |H(e^{-j\omega})|$

where $|H(e^{j\omega})|$ – Magnitude Response – Even fn of $\omega$

$\theta(\omega) = \angle H(e^{j\omega})$ – Phase Response – Odd fn of $\omega$

$\theta(\omega) = -\theta(-\omega)$

- Freq. Response is periodic with period $\omega$.

Frequency Response of first-order system:

The difference equation of $N^{th}$ order system

$$y(n) = \sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k).$$

For the first order system

$$y(n) - a y(n-1) = x(n)$$

By taking fourier transform on both sides

$$Y(e^{j\omega}) - a e^{-j\omega} Y(e^{j\omega}) = X(e^{j\omega})$$

$$Y(e^{j\omega})[1 - a e^{-j\omega}] = X(e^{j\omega})$$

The freq response of first order system

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-ae^{-j\omega}}$$

The impulse response

$$h(n) = F^{-1}[H(e^{j\omega})] = F^{-1}\left[\frac{1}{1-ae^{-j\omega}}\right]$$

$$= a^n u(n)$$

$$H(e^{j\omega}) = \frac{1}{1-a(\cos\omega - j\sin\omega)}$$

$$= \frac{1}{1-a\cos\omega + aj\sin\omega}$$

The Magnitude response

$$|H(e^{j\omega})| = \frac{1}{\sqrt{(1-a\cos\omega)^2 + (a^2\sin^2\omega)}}$$

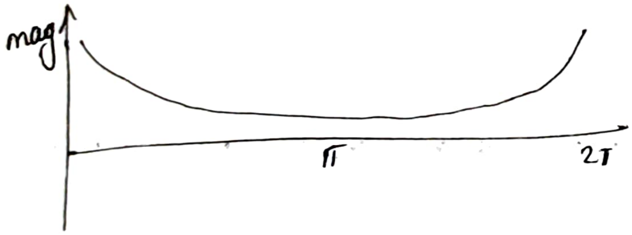
The Phase response

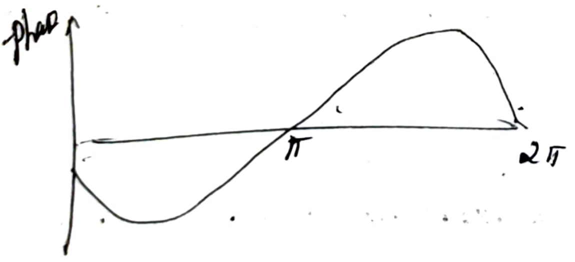
$$\angle H(e^{j\omega}) = -\tan^{-1}\frac{a\sin\omega}{1-a\cos\omega}.$$

$\omega$ — 0 to $2\pi$

For $a = 0.8$     $y(n) = a^n u(n)$   $a = 0.8$

| $\omega$ | 0 | $\pi/4$ | $\pi/2$ | $3\pi/4$ | $\pi$ | $5\pi/4$ | $3\pi/2$ | $7\pi/4$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $H(e^{j\omega})$ $\lvert H(e^{j\omega})\rvert$ | 5 | 1.402 | 0.78 | 0.6 | 0.55 | 0.6 | 0.78 | 1.402 | 5 |
| $\angle H(e^{j\omega})$ | $0^\circ$ | $-52.48^\circ$ | $-38.66^\circ$ | $-19.86^\circ$ | 0 | $19.86^\circ$ | $38.66^\circ$ | $52.48^\circ$ | 0 |




Note:
If $H(e^{j\omega}) < 0$ then angle shd be added with $\pi$.
If $H(e^{j\omega}) > 0$ then angle remains same.

Determine & sketch the magnitude & phase response of

$y(n) = \frac{1}{2}[x(n) + x(n-2)]$.

Sol

Given $y(n) = \frac{1}{2}[x(n) + x(n-2)]$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} [x(n) + x(n-2)]\, e^{-j\omega n}$$

$$= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} + \sum_{n=-\infty}^{\infty} x(n-2)\, e^{-j\omega n}\right]$$

$$= \frac{1}{2}\left[X(e^{j\omega}) + e^{-2j\omega} X(e^{j\omega})\right]$$

$$= \frac{X(e^{j\omega})}{2}\left[1 + e^{-2j\omega}\right]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1+e^{-2j\omega}}{2} = \frac{1+\cos 2\omega - j\sin 2\omega}{2}$$

$|H(e^{j\omega})| = \sqrt{\text{Real Part}^2 + \text{Img Part}^2}$

$$|H(e^{j\omega})| = \frac{1}{2}\sqrt{(1+\cos 2\omega)^2 + \sin^2 2\omega}$$

$$= \frac{1}{2}\sqrt{1 + \cos^2 2\omega + 2\cos 2\omega + \sin^2 2\omega}$$

$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{I_m}{R_e}\right)$

$$= \frac{1}{2}\sqrt{2 + 2\cos 2\omega}$$

$$= \frac{1}{2}\sqrt{2(1+\cos 2\omega)}$$

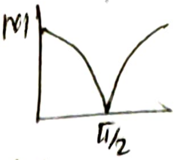
$$= \frac{1}{2}\sqrt{2 \cdot 2\cos^2\omega}$$

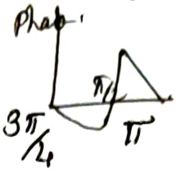
$$= \frac{2}{2}\sqrt{\cos^2\omega} = \cos\omega$$

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{-\sin 2\omega}{1+\cos 2\omega}\right) = \tan^{-1}\left(-\frac{2\sin\omega\cos\omega}{2\cos^2\omega}\right)$$

$$= \tan^{-1}(-\tan\omega) = -\omega$$

$$\left( \begin{aligned} \angle H(e^{j\omega}) &= -\omega \quad \text{for } H(e^{j\omega}) > 0 \\ &= -\omega + \pi \quad \text{for } H(e^{j\omega}) < 0 \end{aligned} \right)$$




| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\omega$ | 0 | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $3\pi/4$ | $\pi$ |
| $H(e^{j\omega})$ | 1 | 0.812 | 0.707 | 0.5 | 0 | $-0.5$ | $-0.707$ | $-1$ |
| $\lvert H(e^{j\omega})\rvert$ | 1 | 0.812 | 0.707 | 0.5 | 0 | 0.5 | 0.707 | 1 |
| $\angle H(e^{j\omega})$ | 0 | $-\pi/6$ | $-\pi/4$ | $-\pi/3$ | $-\pi/2$ | $\pi/3$ | $\pi/4$ | $0°$ |

A discrete-time system has a unit sample response $h(n)$ given by $h(n) = \frac{1}{2}\delta(n) + \delta(n-1) + \frac{1}{2}\delta(n-2)$. Find the system freq. response $H(e^{j\omega})$. Plot magnitude & phase response.

Sol

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2}\delta(n) + \delta(n-1) + \frac{1}{2}\delta(n-2)\right] e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta(n)\, e^{-j\omega n} + \cancel{\tfrac{1}{2}}\sum_{n=-\infty}^{\infty} \delta(n-1)\, e^{-j\omega n} + \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta(n-2)\, e^{-j\omega n}$$

$$= \frac{1}{2}(1) + \cancel{\tfrac{1}{2}}e^{-j\omega} + \frac{1}{2}e^{-2j\omega}$$

$\delta(n) = 1$ for $n=0$

$\delta(n-1) = 1$ for $n=1$

$\delta(n-2) = 1$ for $n=2$.

$$= e^{-j\omega}\left[\frac{1}{2}e^{j\omega} + 1 + \frac{1}{2}e^{-j\omega}\right]$$

$$= e^{-j\omega}(1 + \cos\omega)$$

$$|H(e^{j\omega})| = |e^{-j\omega}|\,|1+\cos\omega| \qquad \because |e^{-j\omega}| = 1$$

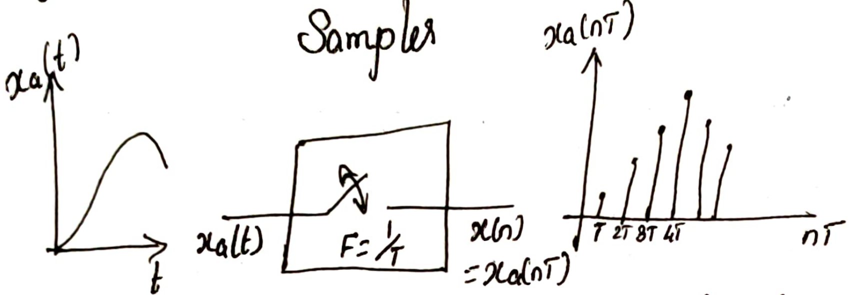
$$= 1 + \cos\omega.$$

$\angle H(e^{j\omega}) = -\omega$ for $0 \leq \omega < \pi$ ($\because H(e^{j\omega})$ is positive for $0 \leq \omega \leq \pi$).

| | | | | | | |
|---|---|---|---|---|---|---|
| $\omega$ | 0 | $\pi/4$ | $\pi/3$ | $\pi/2$ | $3\pi/4$ | $\pi$ |
| $H(e^{j\omega})$ $\lvert H(e^{j\omega})\rvert$ | 2 | 1.707 | 1.5 | 1 | 0.293 | 0 |
| $\angle H(e^{j\omega})$ | 0 | $-\pi/4$ | $-\pi/3$ | $-\pi/2$ | $-3\pi/4$ | $-\pi$ |

## Sampling:

A process of converting a continuous-time signal into a discrete-time signal.



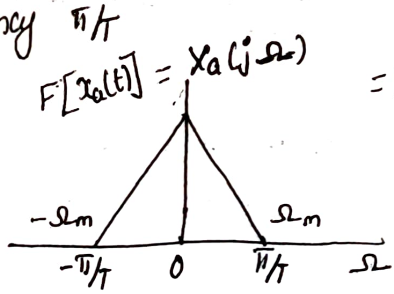

Time interval T between successive samples is called sampling period (or) sampling interval & its reciprocal $1/T = F_s$ is called the sampling rate or the frequency.

## Aliasing Effect:

- The sequence $x(n)$ obtained by sampling $x_a(t)$
- No information loss in sampling then it is possible to recover the continuous-time signal from the samples.

To determine the condition under which no information loss

- Let $x_a(t)$ to be band limited signal with max. frequency $\pi/T$




$F[x_a(t)] = X_a(j\Omega)$ - spectrum of $x_a(t)$

or

~~$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$~~

$\Omega$ - Continuous time freq.

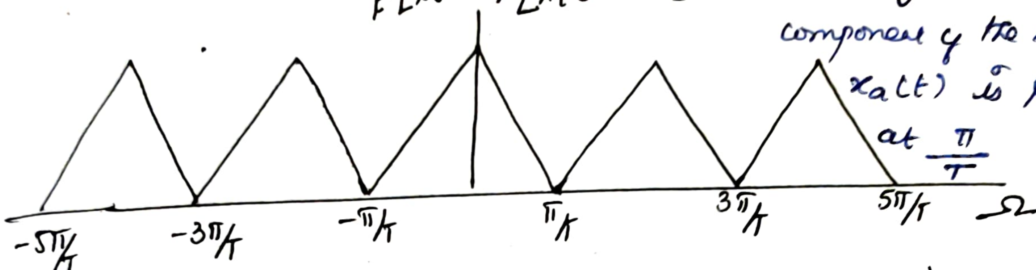
- The freq spectrum of sampled version of the signal for $\Omega = \pi/T$

Analog signal sampled at a freq $f_s = \frac{\pi}{T}$

- The freq spectrum of sampled version with a sampling freq $\Omega$ results in a periodic repetition of $x_a(j\Omega)$ with period $\Omega$ where $\Omega = \pi/T$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$F[x(n)]_T \; [X(e^{j\Omega T})]$ Case i, Highest freq. component of the signal $x_a(t)$ is present at $\frac{\pi}{T}$

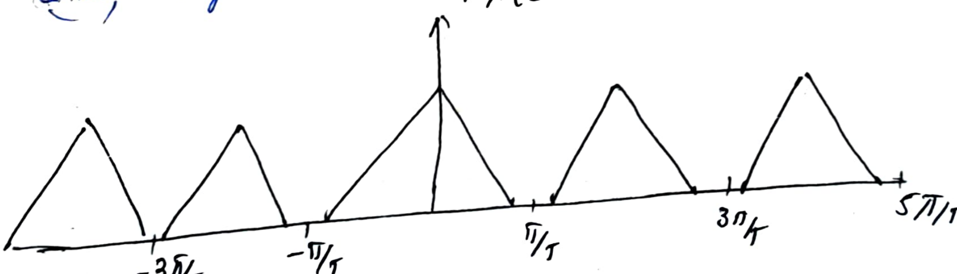



- In this case the spectrum does not overlap. The spectrum $X_a(j\Omega)$ can be recoverd from $X(e^{j\Omega T})$ by using a low pass filter which has sharp cut off at $\Omega = \pi/T$.

- Same is the case if $\Omega < \pi/T$.

Case ii, Highest freq component of the signal $x_a(t) < \frac{\pi}{T}$

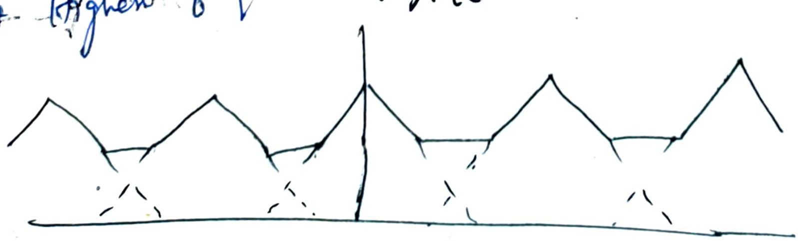
$T X(e^{j\Omega T})$




- If $x_a(t)$ contains the frequency greater than $\pi/T$ overlaps.

Case iii. Highest freq component of the signal $x_a(t) > \frac{\pi}{T}$

$T X(e^{j\Omega T})$



- The superposition of high freq component on the low frequency is known as freq aliasing.

- Because of aliasing, the spectrum $X_a(j\Omega)$ is no longer recoverable by low pass filtering from the spectrum of $X(e^{j\Omega T})$.

- The aliasing error can be prevented if the highest freq component $\Omega_m$ in the signal is less than or equal to $\pi/T$ ie

$$\Omega_m \leq \pi/T$$

- If sampling freq $F = \frac{1}{T}$  $F = f_s$

$$\Omega_m \leq \pi F \quad \cancel{\Omega_s/2}$$

$$2\pi f_m \leq \pi F \quad \text{or} \quad F \geq 2f_m$$

Sampling Theorem: $2\pi f_m \leq \pi f_s$ $\boxed{f_s \geq 2f_m}$

Sampling frequency must be at least twice the highest frequency present in the signal or greater than the signal

$$F \geq 2f_m.$$

Multirate sampling processing:

The discrete time systems that process data at more than one sampling rate are known as Multirate system.

Uses:
- High quality data acquisition & storage systems
- Audio signal processing. Eg. CD is sampled at 44.1kHz but digital audio tape (DAT) is sampled at 48kHz. Conversion b/w DAT & CD uses multirate signal processing tech.

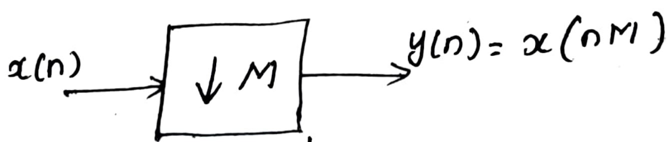
- PAL & NTSC run at different sampling rates. ∴ To watch an American program in Europe, one needs a sampling rate converter.
- In speech processing to reduce the storage space or transmission rate of speech data.
- In transmultiplexers.
- Narrowband filtering for fetal ECG & EEG.

Two basic operations in multirate signal processing

1. Decimation (Down sampling)
2. Interpolation (Up sampling)

Down Sampling:

The sampling rate of a discrete-time signal $x(n)$ can be reduced by a factor M by taking every $M^{th}$ value of the signal.




Down Sampler

Eg:

If $x(n) = \{1, -1, 2, 4, 0, 3, 2, 1, 5 \ldots\}$ then $y(n) = x(nM)$ for $M = 2$. Find the output $y(n)$.

Sol

To down sample by the factor M, $M-1$ samples in between samples of $x(n)$ are left. This process is equal to reducing the sampling rate by a factor M.

$\therefore y(n) = \{1, 2, 0, 2, 5, \ldots\}$

$M = 2$ ; $M - 1 = 1$

## Aliasing Effect:

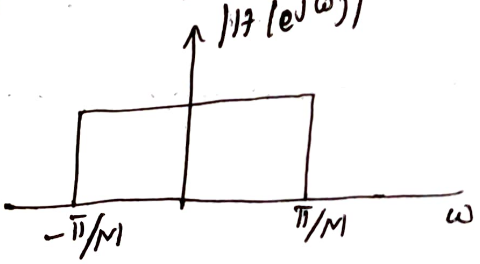
Spectra obtained after down sampling will overlap if the original spectrum is not bandlimited to $\omega = \frac{\pi}{M}$. This overlap causes aliasing.

If the signal $x(n)$ is not band limited to $\pi/M$ then a low pass filter with cut off freq $\pi/M$ is used prior to down sampler called anti-aliasing filter. The complete process of filtering & down sampling is referred as decimation.

Decimation

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow \boxed{\downarrow M} \longrightarrow y(n)$$

Magnitude Response of filter $h(n)$




## Upsampler:

The sampling rate of a discrete time signal can be increased by a factor L by placing L-1 equally spaced zero's b/w each pair of samples.

$$x(n) \longrightarrow \boxed{\uparrow L} \longrightarrow y(n) = x\left(\frac{n}{L}\right)$$

$$y(n) = \begin{cases} x(n/L) & n = 0, \pm L, \pm 2L \\ 0 & \text{otherwise} \end{cases}$$

Eg: If $x(n) = \{1, 2, 4, -2, 3, 2, 1 \ldots\}$ obtain $x(n/2)$.

$L = 2 \quad L-1 = 1$

$y(n) = x\left(\frac{n}{2}\right) = \{1, 0, 2, 0, 4, 0, -2, 0, 3, 0, 2, 0, 1, 0, \ldots\}$

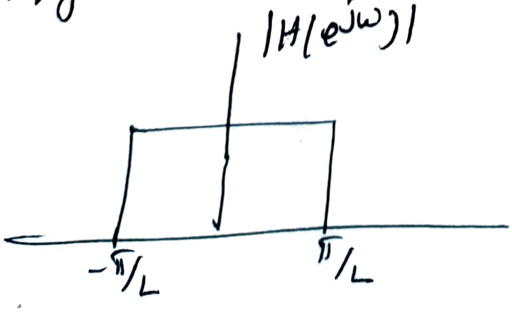
## Aliasing Effect

Frequency spectrum of upsampled signal $y(n)$ by a factor $L$ contain $L-1$ additional images of the input spectrum. These are due to the addition of $L-1$ zero samples b/w successive samples of $x(n)$. Since the image spectrums are not interested, a low pass filter with cut off freq $\omega_c = \pi/L$ can be used after up sampler, used to remove the image spectrum. The complete process of upsampling is filtering is known as interpolation

Interpolation

$$x(n) \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

Magnitude Response of the filter



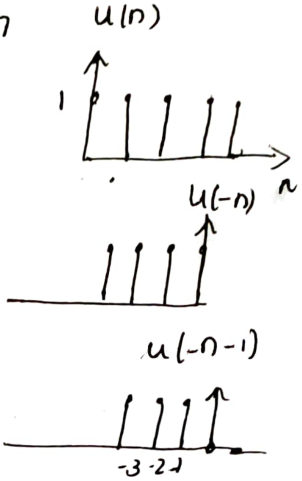
1. Determine the z transform of the signal

$$x(n) = -b^n u(-n-1).$$

Sol

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= -\sum_{n=-\infty}^{\infty} b^n u(-n-1) z^{-n}$$




$$= -\sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} b^{-n} z^n$$

$$= -\sum_{n=1}^{\infty} (b^{-1}z)^n$$

$$= -\frac{b^{-1}z}{1-b^{-1}z}.$$

2. Find the z transform of the following sequence.

i) $x(n) = u(n)$ ii) $x(n) = \delta(n)$ iii) $x(n) = \left(\frac{1}{2}\right)^n u(-n)$.

i) Sol

$$X(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}.$$

$$= \sum_{n=0} (z^{-1})^n$$

$$= \frac{1}{1-z^{-1}}$$

ii) $x(n) = \delta(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

$$= 1$$

iii) $x(n) = \left(\frac{1}{2}\right)^n u(-n)$.

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(-n) z^{-n}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{-n} z^{n}$$

$$= \sum_{n=0}^{\infty} (2z)^n$$

$$= \frac{1}{1-2z}$$

3. z transform $n\,u(n)$.

$$Z\{u(n)\} = \frac{z}{z-1}$$

$$Z\{n\,x(n)\} = -z \frac{d}{dz} X(z).$$

$$= -z \frac{d}{dz}\left(\frac{z}{z-1}\right).$$

$$= \frac{z}{(z-1)^2}$$

1. Z transform of $nu(n)$

$$Z\{u(n)\} = \frac{z}{z-1}$$

$$Z\{nu(n)\} = -z\frac{d}{dz}X(z)$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$$ $\frac{udv - vdu}{v^2}$

$$= -z\left[\frac{z-(z-1)}{(z-1)^2}\right]$$

$$= -z\left[\frac{-1}{(z-1)^2}\right]$$

$$= \frac{z}{(z-1)^2}.$$

2. $x(n) = a^{n-1}u(n-1)$ $\frac{1}{1-az^{-1}} \times \frac{z}{z-a}$

$x(n) = a^n u(n)$

$$X(z) = \frac{1}{1-az^{-1}} \quad \times z.$$

By time shifting property

$$Z\{x(n-k)\} = z^{-k}X(z)$$

$$Z\{a^{n-1}u(n-1)\} = \frac{z^{-1}}{1-az^{-1}} = \frac{1}{z-a}$$

3. $x(n) = r^n \sin\omega_0 n\, u(n)$

$$Z\{\sin\omega_0 n\, u(n)\} = \frac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$$

By using the multiplication by an exponential seq. property

$$Z\{a^n x(n)\} = X(a^{-1} z)$$

$$Z\{r^n \sin(\omega_0 n)\, u(n)\} = \frac{(\sin\omega_0)(r^{-1}z)^{-1}}{1 - 2\cos\omega_0 (r^{-1}z)^{-1} + (r^{-1}z)^2}$$

$$= \frac{r \sin\omega_0 z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-1}}$$

## Inverse Z transform

1. Long division method
2. Partial fraction method
3. Residue method
4. Convolution method.

Long division $\quad X(z) = \dfrac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$

Causal

$$1-2z^{-1}+z^{-2} \;\Big)\; 1+2z^{-1} \;\Big(\; 1+4z^{-1}+7z^{-2}+10z^{-3}+13z^{-4}+16z^{-5}$$

$1+2z^{-1}$

$1-2z^{-1}+z^{-2}$

$\quad + \quad (-)$

$4z^{-1}-z^{-2}$

$4z^{-1}-8z^{-2}+4z^{-3}$

$- \quad (+) \quad (-)$

$7z^{-2}-4z^{-3}$

$7z^{-2}-14z^{-3}+7z^{-4}$

$10z^{-3}-7z^{-4}$

$10z^{-3}-20z^{-4}+10z^{-5}$

$13z^{-4}-10z^{-5}$

$13z^{-4}-26z^{-5}+13z^{-6}$

$16z^{-5}-13z^{-6}$

$16z^{-5}-32z^{-6}+16z^{-7}$

$19z^{-6}-16z^{-7}$

$$x(n) = \{1, 4, 7, 10, 13, 16 \ldots\}$$

$$X(z) = \frac{1+3z^{-1}+7z^{-2}+10z^{-3}+10z^{-4}+7z^{-5}+2z^{-6}}{1+2z^{-1}+3z^{-2}+2z^{-3}}$$

$$X(z) = 1+z^{-1}+2z^{-2}+z^{-3}$$

$$x(n) = \{\underset{\uparrow}{1}, 1, 2, 1\}$$

1. Find the inverse Z transform of the following.

$$X(z) = \frac{\frac{1}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

× by $z^2$

$$X(z) = \frac{\frac{1}{4}z}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

$$\frac{X(z)}{z} = \frac{c_1}{z-\frac{1}{2}} + \frac{c_2}{z-\frac{1}{4}}$$

$$= \frac{1}{z-\frac{1}{2}} - \frac{1}{z-\frac{1}{4}}$$

$$x(z) = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}$$

$$z^{-1}\left\{\frac{z}{z-a}\right\} = a^n u(n).$$

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n).$$

2. Find the Inverse z-transform of the following

$$X(z) = \frac{z+4}{z^2-4z+3}$$

$$= \frac{z+4}{(z-1)(z-3)} = \frac{A}{(z-1)} + \frac{B}{(z-3)}$$

$$X(z) = \frac{A}{(z-1)} + \frac{B}{(z-3)}$$

$$X(z) = \frac{-5}{2(z-1)} + \frac{7}{2(z-3)}$$

$$\Rightarrow \quad x(n) = \frac{-5}{2}u(n-1) + \frac{7}{2}(3)^{n-1}u(n-1)$$

$$z^{-1}\left[\frac{1}{z-1}\right] = u(n-1)$$

$$z^{-1}\left[\frac{1}{z-3}\right] = 3^{n-1}u(n-1)$$

## Unit II

### Definition of the z-transform:

The z-transform of a discrete-time signal $x(n)$ is defined as

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$ . Referred to as two-side z transform.

where $z$ is a complex variable.

In polar form $z$ can be expressed as

$$z = re^{j\omega}$$

where $r$ is the radius of the circle.

Substituting the value of $z$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n}$$

For $r=1$, the expression reduces to fourier transform of $x(n)$. ie z-transform evaluated on the unit circle corresponds to the FT.

If $x(n)$ is a causal sequence ie $x(n)=0$ for $n<0$, then the z-transform is

$$X_{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$ Referred to as one-side z transform.

### Region of convergence:

The set of $z$ values for which the sum converges define a region in z-plane referred to as 'region of convergence (or) ROC'.

The inverse z transform of $X(z)$ is

$$x(n) = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$$

## Properties of Z-transform:

**Linearity:**

If $X_1(z) = Z\{x_1(n)\}$ & $X_2(z) = Z\{x_2(n)\}$ then

$$Z\{a x_1(n) + b x_2(n)\} = a X_1(z) + b X_2(z)$$

**Proof**

$$Z\{a x_1(n) + b x_2(n)\} = \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= a X_1(z) + b X_2(z)$$

**Time shift or translation:**

a) If $X(z) = Z\{x(n)\}$ & the initial conditions for $x(n)$ are zeros, then

$$Z\{x(n-m)\} = z^{-m} X(z).$$

where $m$ is a positive or negative integer.

**Proof**

$$Z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n} = z^{-m} \sum_{n=-\infty}^{\infty} x(n-m) z^{-(n-m)}$$

Let $(n-m) = l$ then

$$Z\{x(n-m)\} = z^{-m} \sum_{l=-\infty}^{\infty} x(l) z^{-l}$$

$$= z^{-m} X(z).$$

b) If $X_+(z) = Z\{x(n)\}$ then

i) $Z\{x(n-m)\} = z^{-m}\left\{X_+(z) + \sum_{k=1}^{m} x(-k) z^{k}\right\}$

ii) $Z\{x(n+m)\} = z^{m}\left\{X_+(z) - \sum_{k=0}^{m-1} x(k) z^{-k}\right\}$

where $m$ is a positive integer.

i) Proof

$$Z\{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m) z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n-m) z^{-(n-m)} \cdot z^{-m}$$

$$= z^{-m} \sum_{n=0}^{\infty} x(n-m) z^{-(n-m)}$$

$$= z^{-m} \sum_{l=-m}^{\infty} x(l) z^{-l} \quad \left[\because\ l = n-m \right]$$

$n = l+m$.

$0 \Rightarrow l = -m$

$\infty \Rightarrow l = \infty$.

$$= z^{-m}\left\{\sum_{l=0}^{\infty} x(l) z^{-l} + \sum_{l=-m}^{-1} x(l) z^{-l}\right\}$$

$$= z^{-m}\left\{X_+(z) + \sum_{k=1}^{m} x(-k) z^{k}\right\}$$

where $l = -k$.

ii) Proof.

$$Z\{x(n+m)\} = \sum_{n=0}^{\infty} x(n+m) z^{-n}$$

$$= z^{m} \sum_{n=0}^{\infty} x(n+m) z^{-(n+m)}$$

$$= z^{m} \sum_{l=m}^{\infty} x(l) z^{-l} \quad \text{where } l = n+m$$

$$= z^{m}\left\{\sum_{l=0}^{\infty} x(l) z^{-l} - \sum_{l=0}^{m-1} x(l) z^{-l}\right\}$$

$$= z^{m}\left\{X_+(z) - \sum_{k=0}^{m-1} x(k) z^{-k}\right\}.$$

Multiplication by an exponential sequence:

If $X(z) = Z\{x(n)\}$, then

$$Z\{a^n x(n)\} = X(a^{-1}z)$$

Proof

$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1}z)^{n}$$

$$= X(a^{-1}z)$$

where the ROC is $|a| R_1 < |z| < |a| R_2$.

Time Reversal

If $X(z) = Z\{x(n)\}$, then

$$Z\{x(-n)\} = X(z^{-1})$$

Proof

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x(l) z^{l} \quad \text{where } l = -n$$

$$= \sum_{l=-\infty}^{\infty} x(l)(z^{-1})^{-l}$$

$$= X(z^{-1})$$

where the ROC is $\frac{1}{R_2} < |z| < \frac{1}{R_1}$.

Differentiation of $X(z)$

If $X(z) = Z\{x(n)\}$, then

$$Z\{n x(n)\} = -z \frac{d}{dz} X(z)$$

Proof

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Differentiating the z transform

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz}(z^{-n})$$

$$= \sum_{n=-\infty}^{\infty} x(n)(-n)\, z^{-n-1}$$

$$= \frac{-1}{z} \sum_{n=-\infty}^{\infty} n\, x(n)\, z^{-n}.$$

$$-z \frac{d}{dz} X(z) = (-z)\left(\frac{-1}{z}\right) \sum_{n=-\infty}^{\infty} n\, x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} n\, x(n)\, z^{-n}$$

$$= Z\{n\, x(n)\}$$

Convolution Theorem:

If $X(z) = Z\{x(n)\}$, & $H(z) = Z\{h(n)\}$, then

$$Z\{x(n) * h(n)\} = X(z)\, H(z).$$

Proof

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$Y(z) = Z\{y(n)\} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, h(n-k)\right] z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)\, z^{-(n-k)} \cdot z^{-k}$$

Interchanging the order of the summation.

$$Y(z) = \sum_{k=-\infty}^{\infty} x(k)\, z^{-k} \cdot \sum_{n=-\infty}^{\infty} h(n-k)\, z^{-(n-k)}.$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, z^{-k} \cdot \sum_{l=-\infty}^{\infty} h(\cancel{n-k}\; l)\, z^{-l} \qquad \because n-k = l.$$

$$= X(z)\, H(z)$$

## Complex Convolution Theorem:

The z-transform of the product of two seq. is related to the z transforms of the individual sequences through the complex convolution theorem. This theorem states that if

$$x_3(n) = x_1(n)\, x_2(n) \text{ then}$$

$$X_3(z) = \frac{1}{2\pi j} \oint_c X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1} dv.$$

The convergence region for $X_3(z)$ consists of all $z$ such that if $v$ is in the region of convergence for $X_1(z)$, then $\frac{z}{v}$ is in the region of convergence for $X_2(z)$. The contour of integration $c$ is a closed contour inside the intersection of the convergence regions for $X_1(v)$ & $X_2(z/v)$.

## Parseval's theorem:

Let us consider two complex sequences $x_1(n)$ & $x_2(n)$. Parseval's relation states that

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint_c X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv.$$

where the contour of integration must be in the overlap of the region of convergence of $X_1(v)$ & $X_2^*\left(\frac{1}{v^*}\right)$.

## Initial value theorem:

If $X_+(z) = Z\{x(n)\}$, then $x(0) = \lim_{z\to\infty} X_+(z)$

Final value theorem:

If $X_+(z) = Z\{x(n)\}$, where the ROC for $X_+(z)$ includes, but is not necessarily confined to $|z|>1$ & $(z-1)X_+(z)$ has no poles on or outside the unit circle

then $$x(\infty) = \lim_{z \to 1} (z-1)X_+(z).$$

Correlation:

If $X_1(z) = Z\{x_1(n)\}$ & $X_2(z) = Z\{x_2(n)\}$ then

$$Z\{r_{x_1x_2}(l)\} = Z\left\{\sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l)\right\}$$

$$= R_{x_1x_2}(z) = X_1(z) X_2(z^{-1}).$$

Digital Filter:

- Is a linear time-invariant discrete-time system.

1. Recursive

A system whose output $y(n)$ at time $n$ depends on any no. of past outputs $y(n-1)$, $y(n-2)$. . . , any inputs $x(n-1)$, $x(n-2)$ . . . . & present input $x(n)$ is called a recursive system.

Eg : IIR

2. Non recursive

A system whose output $y(n)$ at time $n$ depends on present & past inputs then such a system is called non recursive system

Eg : FIR

(*) Structures for the realization of discrete-time systems

Linear time-invariant discrete-time systems are characterized by the general linear constant-coefficient difference eq.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k).$$

By means of the z transform

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}.$$

$$Y(z)\left[1+\sum_{k=1}^{N} a_k z^{-k}\right] = X(z)\sum_{k=0}^{M} b_k z^{-k}$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}.$$

$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \dots}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \dots}$$

Basic IIR digital filter:

An $N^{th}$ order IIR digital filter transfer fn is characterised by $2N+1$ unique coefficients & require $2N+1$ multipliers & $2N$ two-input adders for implementation.

Non Canonical Form: (Direct Form I Structure).

Consider $3^{rd}$ order IIR filter characterised by transfer fn.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

$$Y(z)\left[1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}\right] = X(z)\left[b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}\right]$$

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - a_3 z^{-3} Y(z) + b_0 X(z) + b_1 X(z) z^{-1} + b_2 X(z) z^{-2} + b_3 X(z) z^{-3}$$

taking inverse z transform.

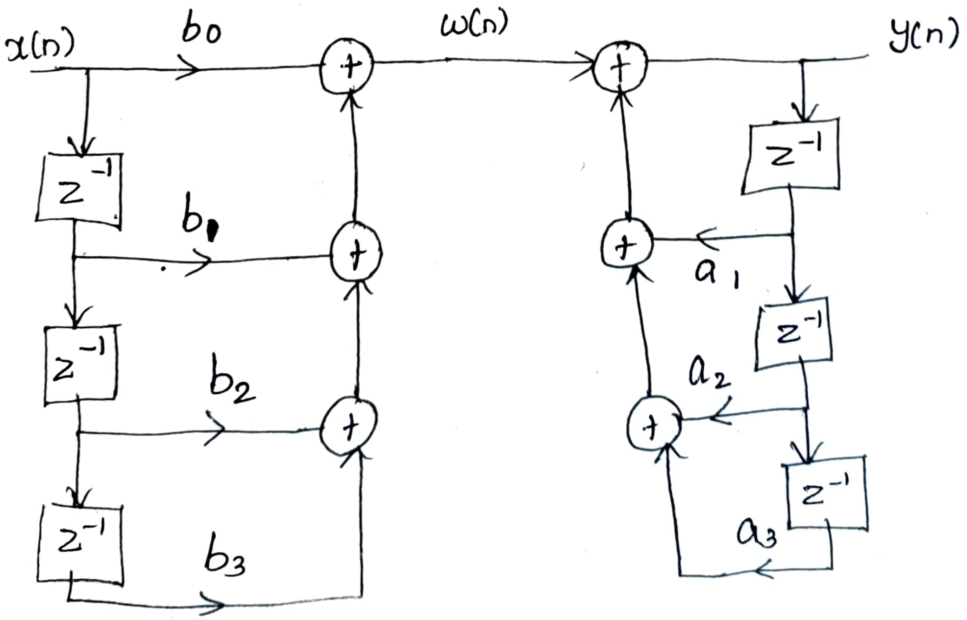
$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3)$$

Let $b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3) = w(n)$

then

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) + w(n).$$

Realization of $w(n)$ & $y(n)$.



Canonical Structure (Direct form II)

Consider 3rd order IIR filter characterised by transfer fn

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

Let

$$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

$$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

$$X(z) = W(z) + a_1 W(z) z^{-1} + a_2 W(z) z^{-2} + a_3 W(z) z^{-3}$$

by taking inverse z transform

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) - a_3 w(n-3) \quad \text{(1)}$$

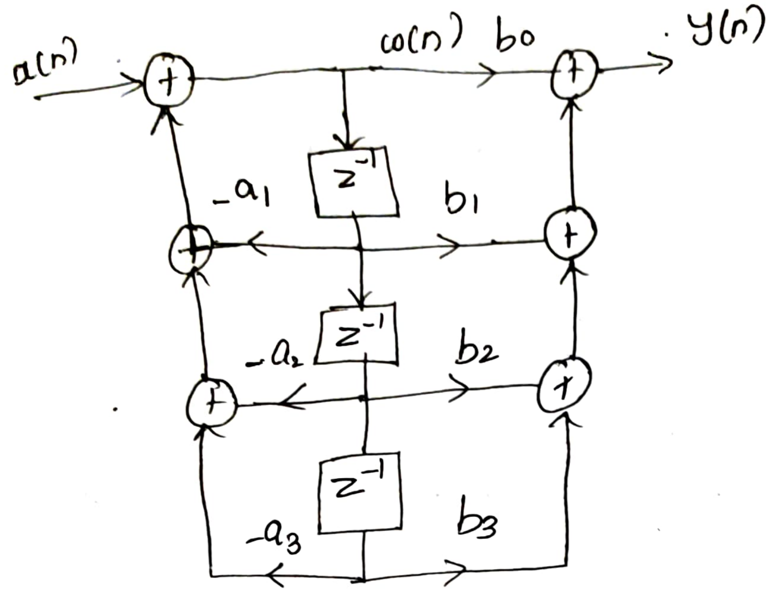
$$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}$$

$$Y(z) = b_0 W(z) + b_1 W(z) z^{-1} + b_2 W(z) z^{-2} + b_3 W(z) z^{-3}$$

by taking inverse z transform

$$y(n) = b_0 w(n) + b_1 w(n-1) + b_2 w(n-2) + b_3 w(n-3) \quad \text{(2)}$$

Realization of $w(n)$ & $y(n)$




Transposed structure:

<u>Signal flow graph</u>: Graphical representation of the relationship b/w the variables of a set of linear difference equations, which is basically a set of directed branches that connect at node.

Branches & nodes are basic elements of signal flow graph.

<u>Node</u>: Represents a system variable, which is equal to the sum of incoming signals from all branches connecting to the node.

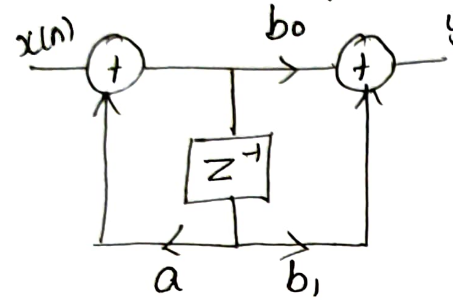
<u>Two Types</u>: <u>Source Node</u>: I/P to the system originates at source node that have no entering branches.

<u>Sink Node</u>: O/P signal is extracted at sink node, that have only entering branches.

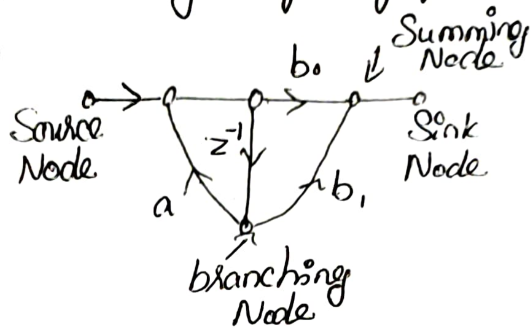
<u>Branch</u>: Signal travels along a branch from node to another node. Arrow head shows the direction of the branch.

Delay: Indicated by the branch transmittance $z^{-1}$

Block diagram representation of $1^{st}$ order digital filter




Signal flow graph




Signal flow graph contains the same basic information as the block diagram realization of the system. Only difference is that both branch point & adders in the block diagram are represented by nodes.

Transposition (or) flow graph reversal theorem:

The system function remains unchanged even if the direction of all branch transmittance are reversed & also the interchange of i/ps o/p in the flow graph.

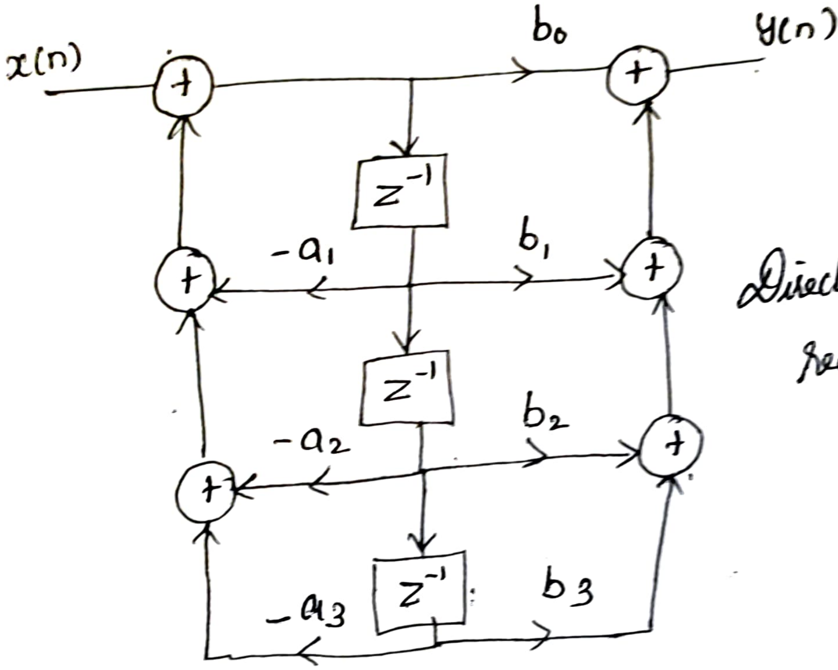
The resultant structure is called a transposed structure.

Steps to find the transposed structure:

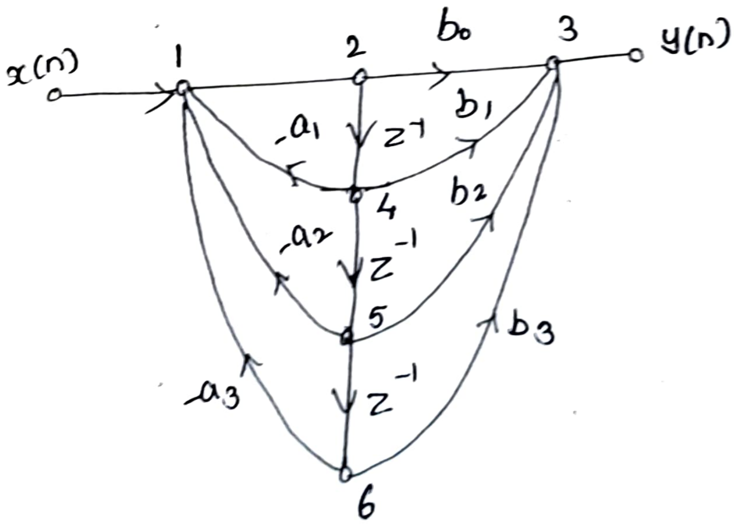
1. Reverse the direction of all branches in the signal flow graph.
2. Interchange the i/ps & o/ps.

3. Reverse the roles of all nodes in the flow graph.
4. Summing points become branching points.
5. Branching points become summing points.

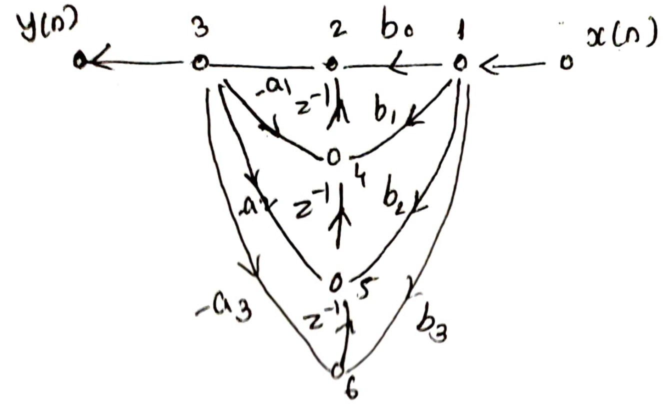
Consider 3rd order IIR filter characterised by transfer function & realize using direct form II



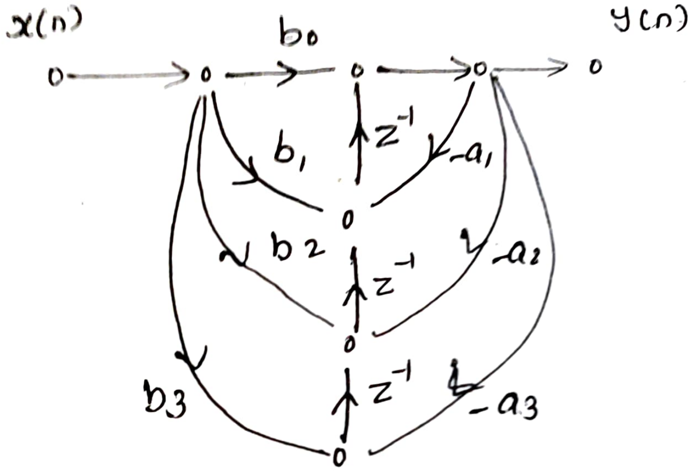

Direct form II realization.



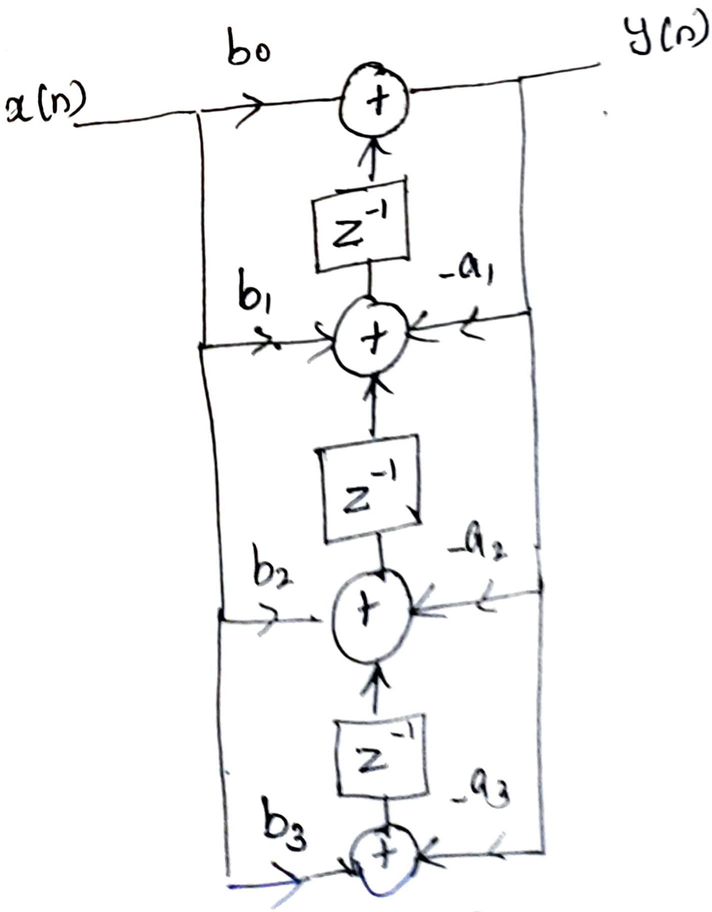

Signal flow graph.



y(n)
3
2
b0
1
x(n)
-a1
z^-1
b1
4
z^-1
b2
5
-a3
z^-1
b3
6

x(n)
b0
y(n)
b1
z^-1
-a1
b2
z^-1
-a2
b3
z^-1
-a3

b0
y(n)
x(n)
+
z^-1
-a1
b1
+
z^-1
-a2
b2
+
z^-1
-a3
b3
+

Cascade Form:

- The system can be factored into cascade of second-order subsystems such that $H(z)$ can be expressed as

$$H(z) = \prod_{k=1}^{k} H_k(z) = H_1(z) \cdot H_2(z) \cdot \ldots \cdot H_k(z)$$

where $k$ - integer part of $\frac{(N+1)}{2}$

where 
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

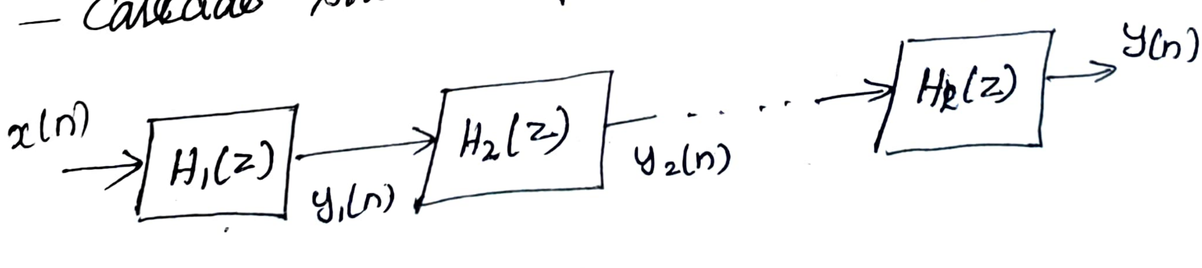
$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \ldots}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \ldots}$$

- The general form of $H_k(z)$

$$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}$$

- Each of the second order subsystems can be realized in either direct form I (or) direct form II (or) transposed form II

- Cascade structure of second-order systems.



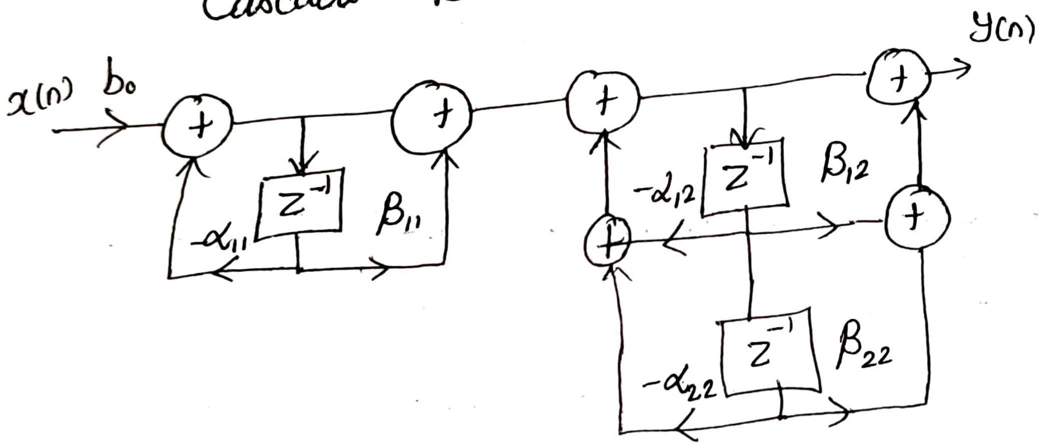
Consider a third order digital filter

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

This can be factorised into a product of first order & second order polynomial as follows.

$$H(z) = b_0 \left[\frac{1+\beta_{11} z^{-1}}{1+\alpha_{11} z^{-1}}\right]\left[\frac{1+\beta_{12} z^{-1} + \beta_{22} z^{-2}}{1+\alpha_{12} z^{-1} + \alpha_{22} z^{-2}}\right]$$

Cascade Realization

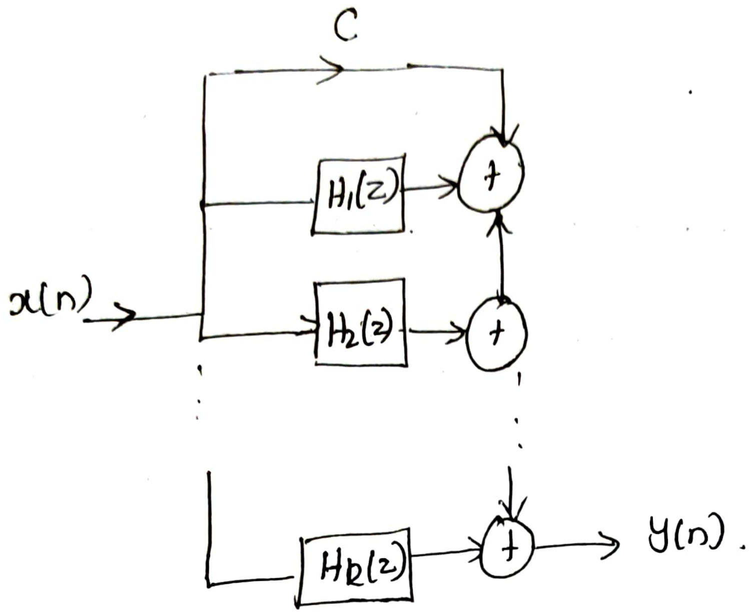



Parallel Realization:

An IIR transfer function can be realised in a parallel form by making use of partial fraction expansion of the transfer function H(z)

$$H(z) = c + \sum_{k=1}^{N} \frac{A_k}{1 - P_k z^{-1}}$$

where $\{P_k\}$ – Poles $\{A_k\}$ – Residues.




**Eg:**

1. Determine the direct form I, direct form II & transposed direct form II for the given system.

$$y(n) = \tfrac{1}{2} y(n-1) - \tfrac{1}{4} y(n-2) + x(n) + x(n-1)$$

**Sol**

Direct form I

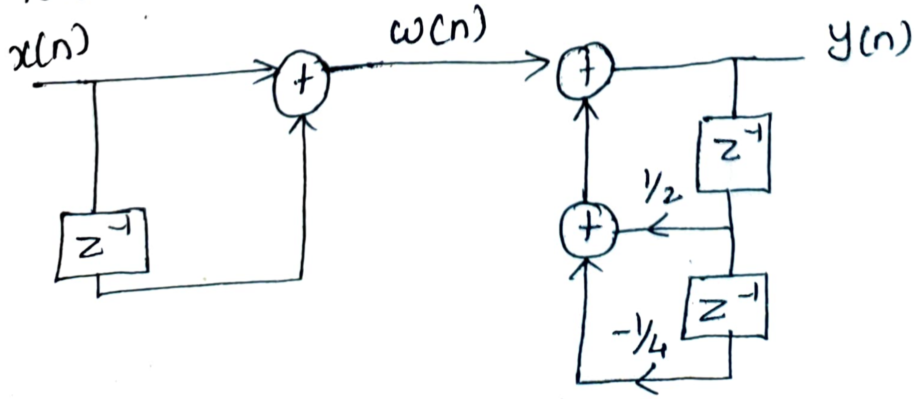
Given

$$y(n) = \tfrac{1}{2} y(n-1) - \tfrac{1}{4} y(n-2) + x(n) + x(n-1)$$

Let $x(n) + x(n-1) = \omega(n)$ — ①

$$\therefore \quad y(n) = \tfrac{1}{2} y(n-1) - \tfrac{1}{4} y(n-2) + \omega(n)$$

Realization of $\omega(n)$ & $y(n)$



Direct form II

Taking z transform

$$Y(z) = \frac{1}{2} z^{-1} Y(z) - \frac{1}{4} z^{-2} Y(z) + X(z) + z^{-1} X(z)$$

$$Y(z)\left[1 - 0.5z^{-1} + 0.25z^{-2}\right] = X(z)\left[1 + z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-0.5z^{-1}+0.25z^{-2}} = 1+z^{-1} \cdot \frac{1}{1-0.5z^{-1}+0.25z^{-2}}$$

$$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \cdot \frac{Y(z)}{W(z)}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1-0.5z^{-1}+0.25z^{-2}}$$

$$W(z) - 0.5z^{-1}W(z) + 0.25z^{-2}W(z) = X(z)$$

$$W(z) = X(z) + 0.5z^{-1}W(z) - 0.25z^{-2}W(z)$$

Taking inverse z transform

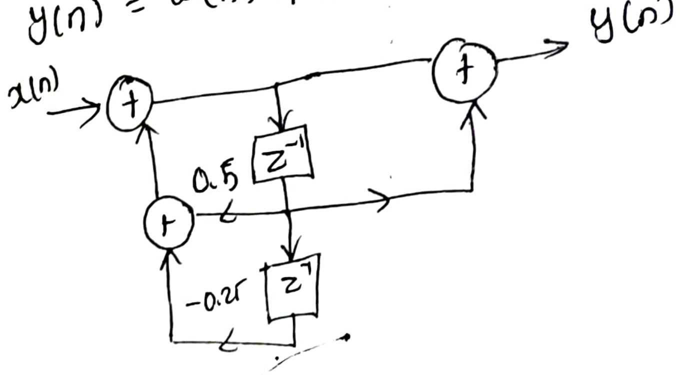
$$w(n) = x(n) + 0.5w(n-1) - 0.25w(n-2) \quad \text{—①}$$

$$\frac{Y(z)}{W(z)} = 1+z^{-1}$$

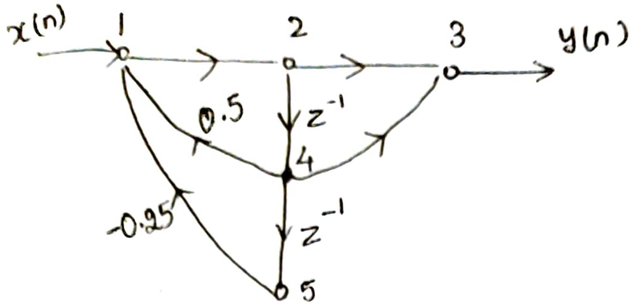
$$Y(z) = W(z) + z^{-1}W(z)$$

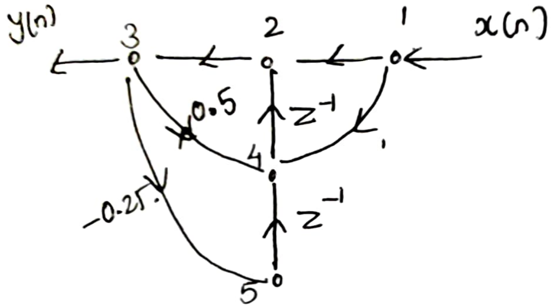
Taking inverse z transform

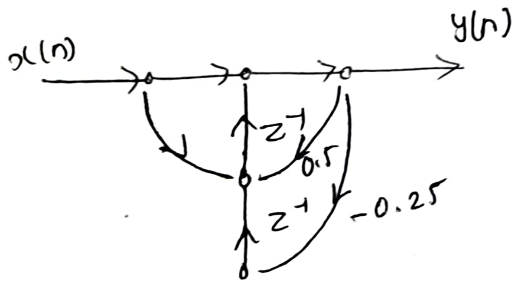
$$y(n) = w(n) + w(n-1)$$

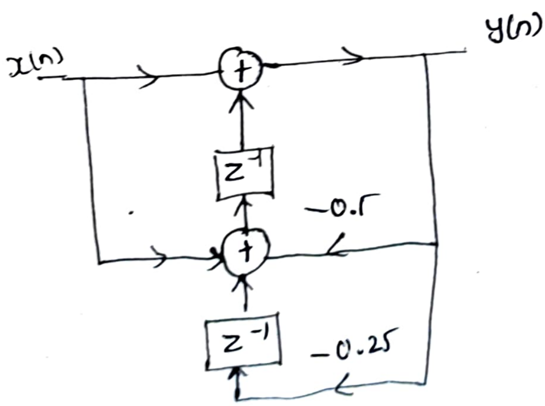


Transposed direct form II



Obtain the cascade & parallel form realization for the system $y(n) = -0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$.

**sol**

Given

$$y(n) = -0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$$

Cascade Form:

Taking z transform

$$Y(z) = -0.1Y(z)z^{-1} + 0.2z^{-2}Y(z) + 3X(z) + 3.6z^{-1}X(z) + 0.6z^{-2}X(z)$$

$$Y(z)[1 + 0.1z^{-1} - 0.2z^{-2}] = X(z)[3 + 3.6z^{-1} + 0.6z^{-2}]$$

1   1.2   0.2

$$\frac{Y(z)}{X(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

- both negative   → any one -ve

$0.5 \times 0.4$

$$= \frac{3(3+0.6z^{-1})(1+z^{-1})}{(1+0.5z^{-1})(1-0.4z^{-1})}$$

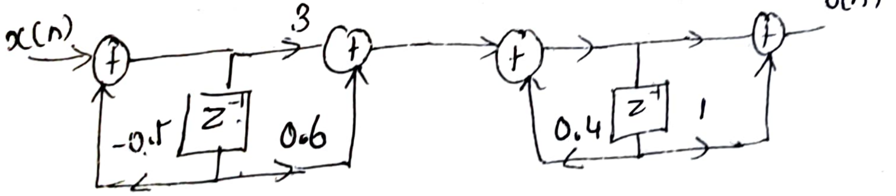
$3(1+1.2z^{-1}+0.2$

$1+z^{-1}$

Let $H_1(z) = \dfrac{3+0.6z^{-1}}{1+0.5z^{-1}}$ & $H_2(z) = \dfrac{1+z^{-1}}{1-0.4z^{-1}}$

Realization of $H_1(z)$ & $H_2(z)$ & cascade to form

Cascade structure




**Problem**

$$H(z) = \frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2})}{(1+\frac{1}{4}z^{-1})(1+z^{-1}+\frac{1}{2}z^{-2})(1-\frac{1}{4}z^{-1}+\frac{1}{2}z^{-2})}$$

Note

$$\frac{A}{(1+\frac{1}{4}z^{-1})} + \frac{Bz^{-1}+C}{1+z^{-1}+\frac{1}{2}z^{-2}} + \dots$$

Parallel Form:

$$H(z) = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

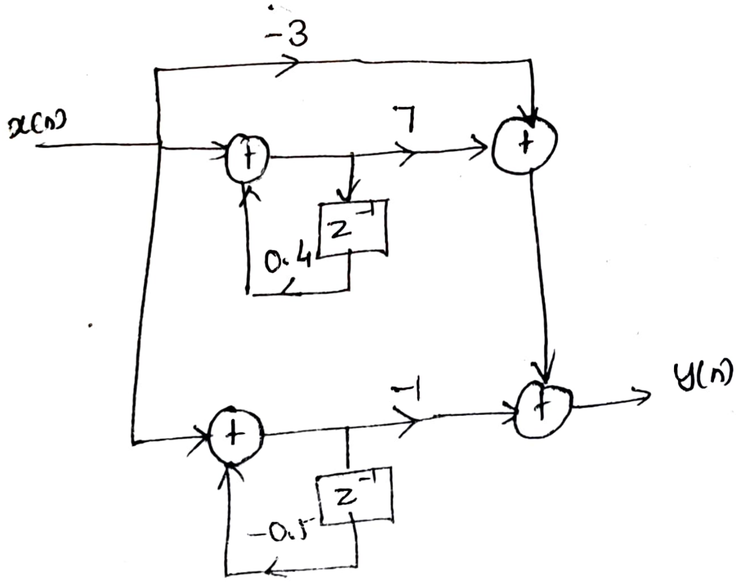
$$\begin{array}{r|l} & -3 \\ -0.2z^{-2} + 0.1z^{-1} + 1 & 0.6z^{-2} + 3.6z^{-1} + 3 \\ & 0.6z^{-2} - 0.3z^{-1} - 3 \\ \hline & 3.9z^{-1} + 6 \end{array}$$

$$\therefore H(z) = -3 + \frac{3.9z^{-1} + 6}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$= -3 + \frac{3.9z^{-1} + 6}{(1 - 0.4z^{-1})(1 + 0.5z^{-1})}$$

$$= -3 + \frac{A}{1 - 0.4z^{-1}} + \frac{B}{1 + 0.5z^{-1}}$$

$$\therefore = -3 + \frac{7}{1 - 0.4z^{-1}} - \frac{1}{1 + 0.5z^{-1}}$$




Obtain the direct form I, direct form II, Transposed cascade & parallel form realization for the system $y(n) = -0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$.

$$H(z) = \frac{(1 + \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2})(1 - \frac{3}{2}z^{-1} + z^{-2})}{(1 + z^{-1} + \frac{1}{4}z^{-2})(1 + \frac{1}{4}z^{-1} + \frac{1}{2}z^{-2})}$$

$y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2)$

## Structure for FIR Systems:

An FIR system is described by the difference equation

$$y(n) = \sum_{k=0}^{N-1} b_k x(n-k) \quad \text{— (1)}$$

by taking z transform

$$Y(z) = \sum_{k=0}^{N-1} b_k z^{-k} X(z)$$

$$\frac{Y(z)}{X(z)} = \sum_{k=0}^{N-1} b_k z^{-k}$$

$$H(z) = \sum_{k=0}^{N-1} b_k z^{-k} \quad \text{— (2)} \rightarrow \underline{h(n) = b_k}$$

The unit sample response of the FIR system is identical to the coefficients $\{b_k\}$

$$h(n) = \begin{cases} b_n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore y(n) = \sum_{k=0}^{N-1} h(k) x(n-k)$$

$$H(z) = \sum_{k=0}^{N-1} h(k) z^{-k}$$

## Transversal Structure: (Direct form)

System function of an FIR filter can be

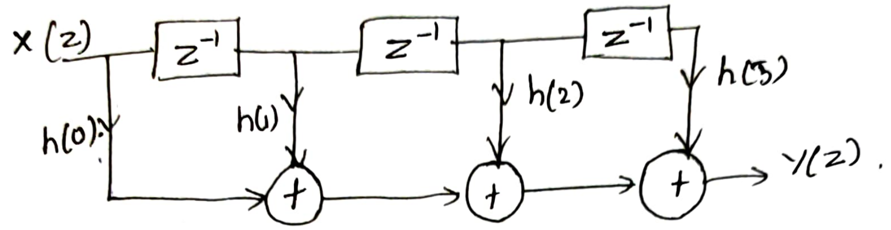
$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

The filter structure in which the multiplier coefficients are precisely the coefficients of the transfer function are called direct form structure.

Realization of FIR filter for $N=4$. Using direct form

$$H(z) = \sum_{n=0}^{4-1} h(n) z^{-n}$$

$$\frac{Y(z)}{X(z)} = h(0) z^{-0} + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3}.$$



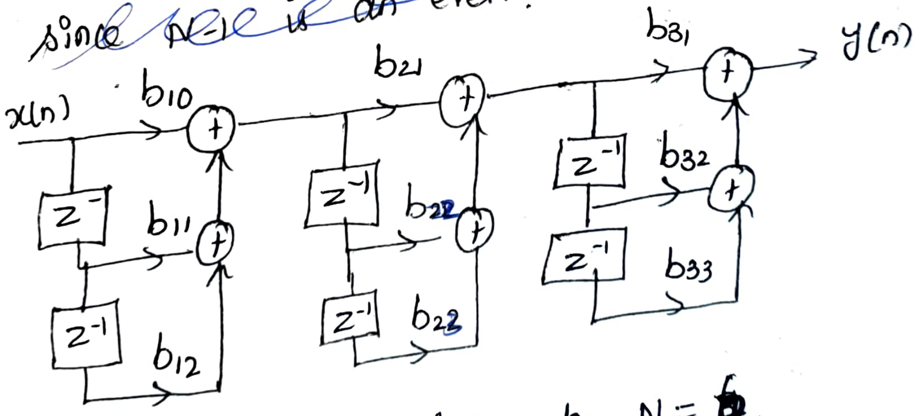

Cascade realization

H(z) can be realized in cascade form from the factored form of H(z)

For N ~~odd~~ even

$$H(z) = \prod_{k=1}^{N/2} (b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2})$$

since ~~N is an even~~.

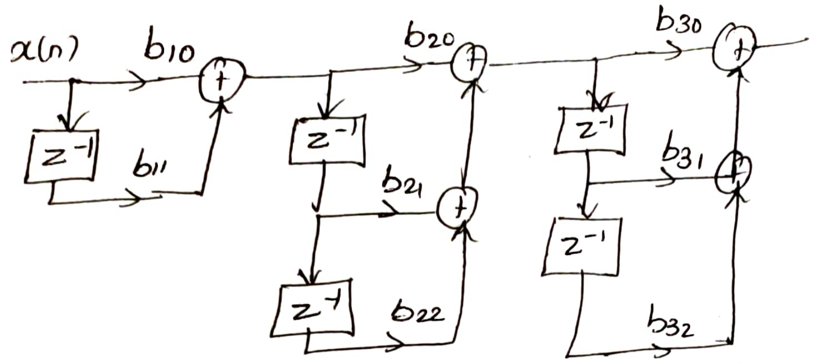



Cascade Realization for $N = 6$.

For N ~~even~~ odd.

$$H(z) = (b_{10} + b_{11} z^{-1}) \prod_{k=1}^{N/2} (b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2})$$

One first order & $\frac{N-1}{2}$ second order.



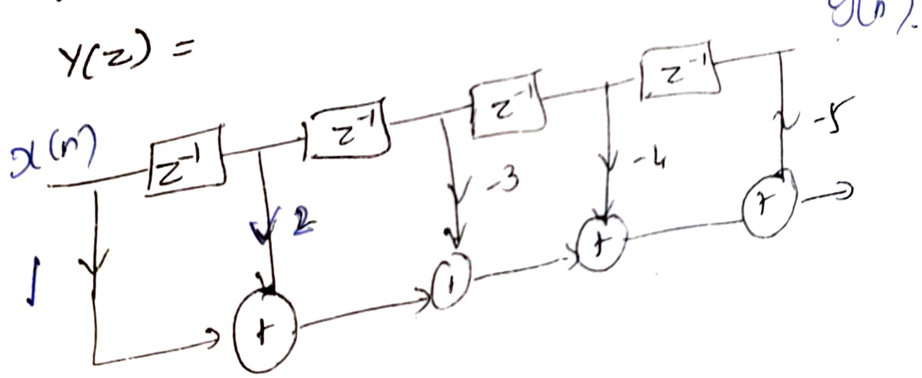
Cascade Realization for $N = 5$

Determine the direct form realization of system function $H(z) = 1+2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}$

Sol

$$\frac{Y(z)}{X(z)} = 1+2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}$$

$Y(z) =$



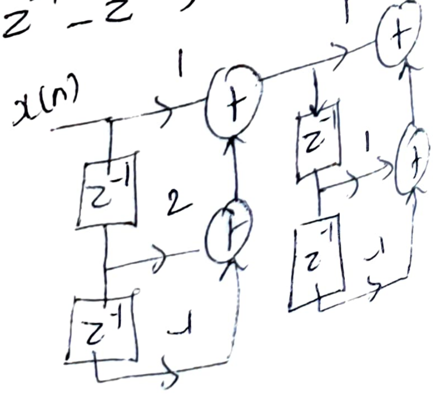
Obtain the cascade realization of system fn $H(z) = (1+ 2z^{-1} - z^{-2})(1+z^{-1} - z^{-2})$

Sol

$H_1(z) = 1+2z^{-1} - z^{-2}$

$H_2(z) = 1+z^{-1} - z^{-2}$

$H(z) = H_1(z) \cdot H_2(z)$



Realize the given impulse response with minimum multipliers $H(z) = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}$.

Sol

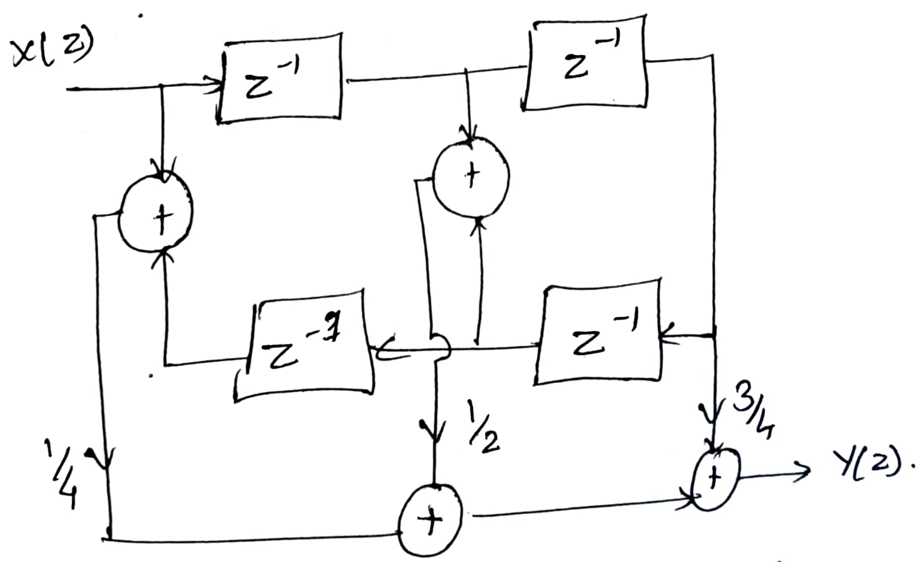
$N = 4. \qquad h(n) = h(N-n)$

$$H(z) = \frac{1}{4}[1 + z^{-4}] + \frac{1}{2}[z^{-1} + z^{-3}] + \frac{3}{4}z^{-2}$$

$$\frac{Y(z)}{X(z)} = \quad ''$$

$$Y(z) = \frac{1}{4}[X(z) + z^{-4}X(z)] + \frac{1}{2}[z^{-1}X(z) + z^{-3}X(z)]$$

$$+ \frac{3}{4}z^{-2}X(z)$$



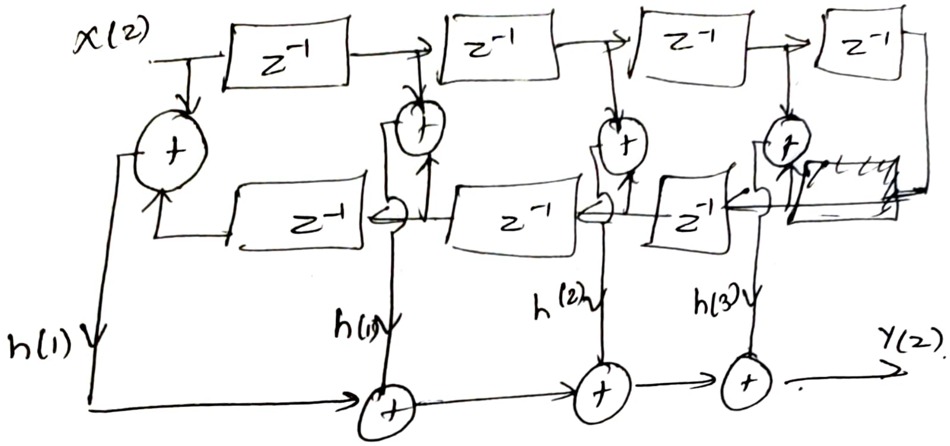

Realize a length 8 type 2 FIR transfer fn.

Sol

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}$$
$$+ h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7}.$$

$$H(z) = h(0)[1 + z^{-7}] + h(1)[z^{-1} + z^{-6}] + h(2)[z^{-2} + z^{-5}]$$
$$+ h(3)[z^{-3} + z^{-4}]$$

$\therefore N = 7 \qquad h(n) = h(N-n)$




Realize the system fn

$$H(z) = \frac{1}{2} + \frac{1}{3} z^{-1} + z^{-2} + \frac{1}{4} z^{-3} + z^{-4} + \frac{1}{3} z^{-5} + \frac{1}{2} z^{-6}.$$

## Unit III

# Design of digital filters

## Design of IIR filters: Analog filter design:

General form of analog filter transfer fn is

$$H(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{M} b_i s^i}{1 + \sum_{i=1}^{N} a_i s^i}$$

where $H(s)$ is the laplace transform of the impulse response $h(t)$. ie

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st}\, dt.$$

- $N \geq M$ must be satisfied
- For a stable analog filter, the poles of $H(s)$ lie in the left half of the s-plane.

There are two types of analog filter design:

1. Butterworth filter
2. Chebyshev filter.

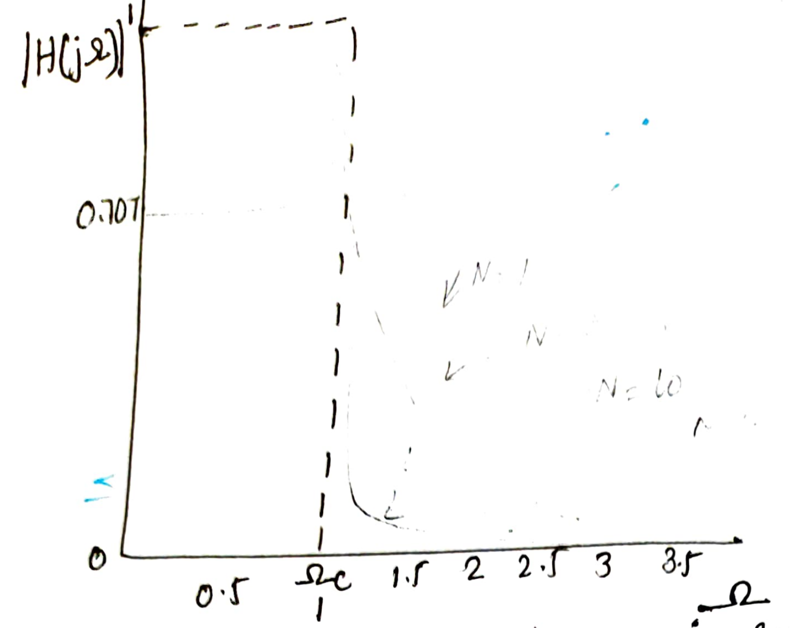
### Analog low pass butterworth filter

The magnitude function of the butterworth low pass filter is given by

$$\boxed{|H(e^{j\Omega})| = \frac{1}{\left[1 + (\Omega/\Omega_c)^{2N}\right]^{1/2}}} \quad N = 1, 2, 3 \ldots$$

where $N$ - order of the filter

$\Omega_c$ - cutoff freq




- Magnitude response approaches the ideal low pass characteristics as the order N increases.

- $\Omega < \Omega_c$   $|H(j\Omega)| \simeq 1$
  $\Omega > \Omega_c$   $|H(j\Omega)|$ decreases rapidly.
  $\Omega = \Omega_c$   curve pass through 0.707, which corresponds to $-3dB$.

Magnitude function of a normalized butterworth filter (cut off freq = 1 rad/sec) as

$$|H(j\Omega)| = \frac{1}{\left[1 + (\Omega)^{2N}\right]^{1/2}}$$

Poles which lies in the left half of the s-plane given by $s_k = e^{j\phi_k}$ where

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2, \dots N$$

unnormalized poles are given by

$$S_k' = \Omega_c S_k$$

Transfer function of butterworth filter for cutoff freq $\Omega_c$ is given by $s \to s/\Omega_c$.

Butterworth polynomial for various values of N for $\Omega_c = 1$ rad/sec.

| N | Denominator of $H(s)$ |
|---|---|
| 1 | $s+1$ |
| 2 | $s^2+\sqrt{2}s+1$ |
| 3 | $(s+1)(s^2+s+1)$ |
| 4 | $(s^2+0.76537s+1)(s^2+1.8477s+1)$ |
| 5 | $(s+1)(s^2+0.618038+1)(s^2+1.618038+1)$ |
| 6 | $(s^2+1.931855s+1)(s^2+\sqrt{2}s+1)(s^2+0.517648s+1)$ |
| 7 | $(s+1)(s^2+1.80194s+1)(s^2+1.247s+1)(s^2+0.445s+1)$ |

Order of the filter is given by

$$\boxed{N \geq \frac{\log\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}}}$$

$$\geq \frac{\log(\lambda/\varepsilon)}{\log\dfrac{\Omega_s}{\Omega_p}}$$

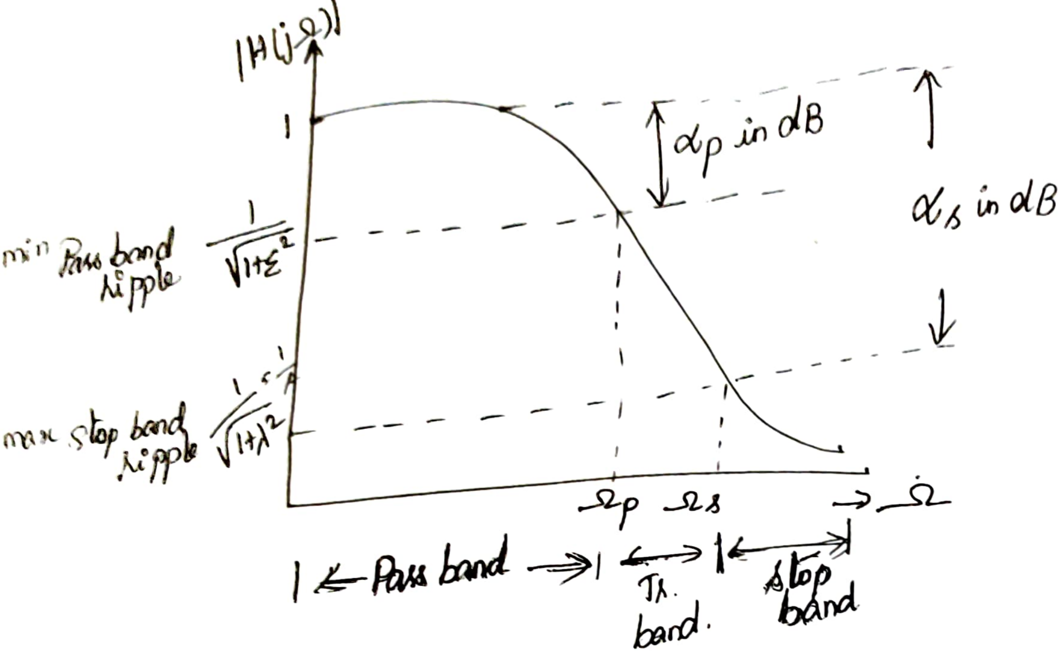
where
$$\boxed{\begin{aligned}\varepsilon &= (10^{0.1\alpha_p}-1)^{0.5}\\ \lambda &= (10^{0.1\alpha_s}-1)^{0.5}\end{aligned}}$$

$$\geq \frac{\log A}{\log(1/k)}$$

$k = \dfrac{\Omega_p}{\Omega_s}$ - transition ratio or selectivity ratio

$A = \lambda/\varepsilon$ -

## Butterworth approximation of magnitude response




Given the specification $\alpha_p = 1 dB$ ; $\alpha_s = 30 dB$ ; $\Omega_p = 200$ rad/sec ; $\Omega_s = 600$ rad/sec. Determine the order of the filter

Sol

$$A = \frac{\lambda}{\varepsilon} = \left(\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right)^{0.5}$$

$$= \left(\frac{10^{3} - 1}{10^{0.1} - 1}\right)^{0.5} = 62.115$$

$$k = \frac{\Omega_p}{\Omega_s} = \frac{200}{600} = 1/3$$ k – transition ratio (or) selectivity ratio

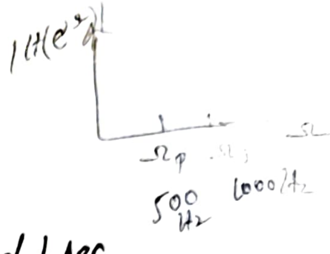
$$N \geq \frac{\log A}{\log(1/k)}$$

$$\geq \frac{\log 62.115}{\log 3} = 3.758$$

Rounding off N to the next higher integer

$N = 4$.

Determine the order of the filter of low pass butterworth filter that has a 3dB attenuation at 500Hz & an attenuation of 40dB at 1000 Hz.




Sol: $\alpha_p = 3dB \qquad \alpha_s = 40dB$

$\Omega_p = 2\pi \times 500 \qquad \Omega_s = 2\pi \times 1000$

$= 1000\pi$ rad/sec. $\qquad = 2000\pi$ rad/sec

$$N \geq \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log \dfrac{\Omega_s}{\Omega_p}} \geq 6.6.$$

$N = 7.$

---

$$S_k' = \Omega_c e^{j\phi_k}$$

$$= 1000\pi\, e^{j\phi_k} \qquad k = 1, 2, 3, \ldots 7$$

where $\phi_k = \dfrac{\pi}{2} + \dfrac{(2k-1)\pi}{2N}$

Steps to design an analog butterworth lowpass filter

1. From the given specifications find the order of the filter N
2. Find the transfer fn H(s) for $\Omega_c = 1$ rad/sec for the value of N
3. calculate the value of cut off freq $\Omega_c$.
4. Find the transfer fn $H_a(s)$ for the above value of $\Omega_c$ by substituting $\boxed{s \rightarrow s/\Omega_c}$ in H(s).

Cut off frequency

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}}$$

Prove that $\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{\Omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}$

sol

The magnitude square function of butterworth analog low pass filter is given by

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

At pass band edge freq.

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

$$1+\left(\frac{\Omega}{\Omega_c}\right)^{2N} = 1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N} = \left(\frac{\Omega}{\Omega_c}\right)^{2N}$$

$$\varepsilon^2\left(\frac{\Omega_c}{\Omega_p}\right)^{2N} = 1$$

$$(\Omega_c)^{\cancel{2N}} = \frac{(\Omega_p)^{\cancel{2N}}}{\varepsilon^2}$$

$$\Omega_c = \frac{\Omega_p}{(\varepsilon^2)^{1/2N}}$$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}}$$

Find the poles of 4th order normalized low pass butterworth filter

Sol

$N = 4 \qquad k = 1, 2, 3, 4$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{8} = \frac{5\pi}{8}; \quad \phi_3 = \frac{\pi}{2} + \frac{5\pi}{8} = \frac{9\pi}{8};$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{8} = \frac{7\pi}{8}; \quad \phi_4 = \frac{\pi}{2} + \frac{7\pi}{8} = \frac{11\pi}{8};$$

$$S_k = e^{j\phi_k}$$

$$S_1 = e^{j5\pi/8} = -0.3827 + j0.9239$$

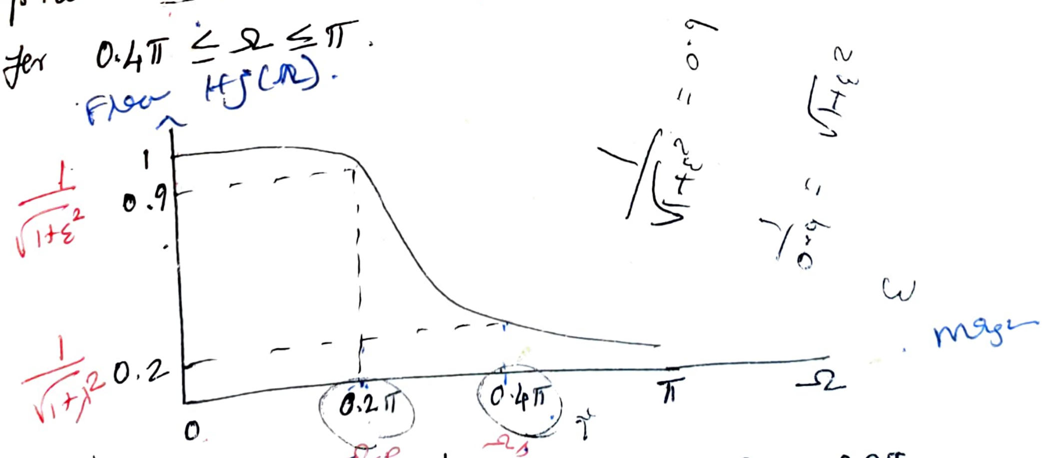
$$S_2 = e^{j7\pi/8} = -0.9239 + j0.3827$$

$$S_3 = e^{j9\pi/8} = -0.9239 - j0.3827$$

$$S_4 = e^{j11\pi/8} = -0.3827 - j0.9239$$

$\Omega s$

For the given specifications design an analog butterworth filter $0.9 \le |H(j\Omega)| \le 1$ for $0 \le \Omega \le 0.2\pi$. $|H(j\Omega)| \le 0.2$ for $0.4\pi \le \Omega \le \pi$.




$$\frac{1}{\sqrt{1+\epsilon^2}} = 0.9 \qquad \frac{1}{\sqrt{1+\lambda^2}} = 0.2 \qquad \Omega_p = 0.2\pi$$

$$\Omega_s = 0.4\pi$$

$$\epsilon = 0.484 \qquad \lambda = 4.898$$

$$N \geq \frac{\log(1/\varepsilon)}{\log\frac{\Omega_s}{\Omega_p}} = \frac{\log\frac{4.\ldots}{0.484}}{\log\left(\frac{0.4\pi}{0.2\pi}\right)} = 3.34$$

$$\boxed{N = 4}$$

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.84776s+1)}$$

$$\boxed{\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}}} = \frac{\Omega_p}{\varepsilon^{1/N}} = \frac{0.2\pi}{(0.484)^{1/4}}$$

$$\Omega_c = 0.24\pi$$

$H(s)$ for $\Omega_c = 0.24\pi$ can be obtained by substituting

$s \to \frac{s}{0.24\pi}$ in $H(s)$ $\quad \boxed{s \to \frac{s}{\Omega_c}}$

$$H(s) = \frac{1}{\left\{\left(\frac{s}{0.24\pi}\right)^2 + 0.76537\left(\frac{s}{0.24\pi}\right)+1\right\}} \times \frac{1}{\left(\frac{s}{0.24\pi}\right)^2 + 1.8477\left(\frac{s}{0.24\pi}\right)+1}$$

$$= \frac{0.323}{(s^2+0.577s+0.0576\pi^2)(s^2+1.393s+0.0576\pi^2)}$$

Design an analog Butterworth filter that has a 2dB passband attenuation at a freq 20 rad/sec & atleast 10dB stopband attenuation at 30 rad/sec.

sol

Given $\alpha_p = 2dB$ $\quad \Omega_p = 20$ rad/sec

$\alpha_s = 10dB$ $\quad \Omega_s = 30$ rad/sec

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$\geq 3.37 \qquad N = 4$

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.84776s+1)}$$

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p}-1\right)^{1/2N}} = \frac{20}{\left(10^{0.2}-1\right)^{1/8}} = 21.3868$$

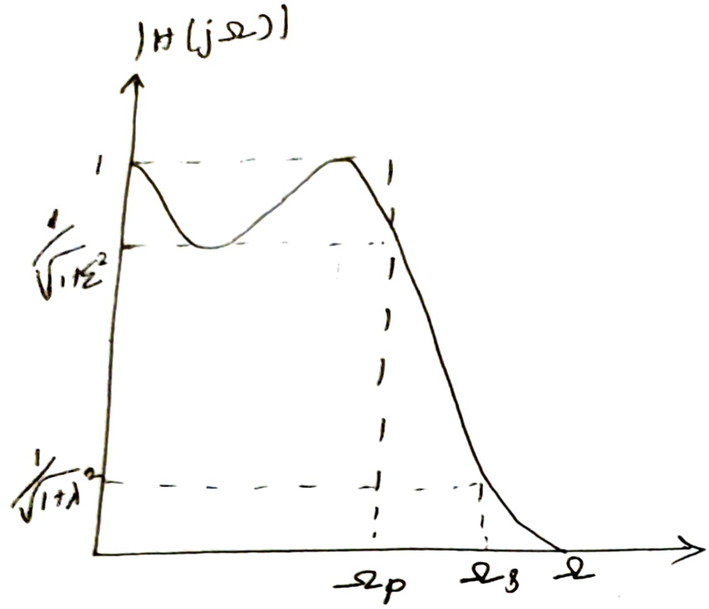
$s \rightarrow s/\Omega_c$.

Analog Low pass Chebyshev filters:

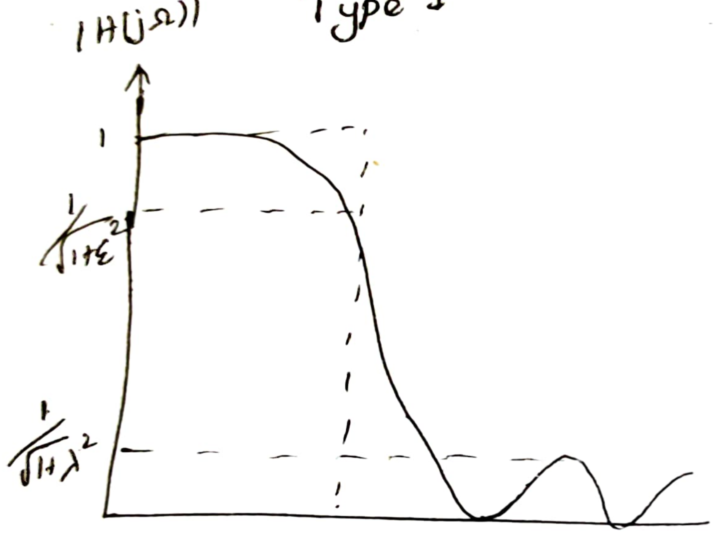
Two types:

Type I Chebyshev filter - All pole filter that exhibit equiripple behaviour in the passband & a monotonic characteristics in the stopband.

Type II Chebyshev filter - Contains both poles & zeros & exhibits a monotonic behaviour in the pass band & an equiripple behaviour in the stop band.




Type I




Type II

The magnitude square response of $N^{th}$ order type I filter can be expressed as

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)} \qquad N = 1, 2, \ldots$$

where $\varepsilon$ – A parameter of the filter related to the ripple in the passband

$C_N(x)$ – $N^{th}$ order Chebyshev polynomial defined as

$$C_N(x) = \cos(N\cos^{-1}x), \quad |x| \le 1 \quad \text{Pass band}$$

&

$$C_N(x) = \cosh(N\cosh^{-1}x), \quad |x| \ge 1 \quad \text{Stop band}$$

The Chebyshev polynomial is defined by the recursive formula

$$C_N(x) = 2x\,C_{N-1}(x) - C_{N-2}(x), \quad N > 1$$

where $C_0(x) = 1$ & $C_1(x) = x$

Order of the filter is defined as

$$N \ge \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

$$N \ge \frac{\cosh^{-1}A}{\cosh^{-1}(1/k)}$$

The poles of a Chebyshev filter can be determined by

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] \quad \text{where } \mu = e^{\sinh^{-1}(\varepsilon^{-1})}$$

$$= \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] \qquad \varepsilon = \sqrt{10^{0.1\alpha_p} - 1}$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2, \ldots N.$$

The poles of the Chebyshev tr. fn are located on an ellipse in the s-plane. The eq. of the ellipse is given by

$$\frac{\sigma_k^2}{a^2} + \frac{\Omega_k^2}{b^2} = 1$$

where a & b are minor & major axes of the ellipse respectively.

$$S_k = a \cos\phi_k + j b \sin\phi_k$$
$$= \sigma_k + j\Omega_k.$$

Given the specifications $\alpha_p = 3dB$; $\alpha_s = 16dB$ $f_p = 1KHz$ & $f_s = 2KHz$. Determine the order of the filter using Chebyshev approximation. Find $H(s)$.

sol

$$\Omega_p = 2\pi \times 1000 = 2000\pi$$
$$\Omega_s = 2\pi \times 2000 = 4000\pi$$
$$\alpha_p = 3dB \qquad \alpha_s = 16dB$$

Step 1:

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}}$$

$$\boxed{= 1.91}$$

$\therefore N = 2$

~~Step 2~~ Note:

N - odd sub. $s=0$ in the denominator poly & find the value, which is equal to the numerator of the transfer fn.

N - even sub. $s=0$ in the denominator. poly & divide the result by $\sqrt{1+\varepsilon^2}$, which is equal to the numerator step 2 :

$\varepsilon = 1, N = 2$

$$\varepsilon = (10^{0.1\alpha_p} - 1)^{0.5} = 1$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 2.414$$

$$a = \Omega_p \left[ \frac{\mu^{1/N} - \mu^{-1/N}}{2} \right] = 910\pi$$

$$b = \Omega_p \left[ \frac{\mu^{1/N} + \mu^{-1/N}}{2} \right] = 2197\pi$$

Step 3 : 200

$$S_k = a\cos\phi_k + jb\sin\phi_k \qquad k = 1, 2.$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad \phi_1 = 135^\circ \quad \phi_2 = 225^\circ$$

$$S_1 = a\cos\phi_1 + jb\sin\phi_1 = -643.46\pi + j1554\pi$$

$$S_2 = a\cos\phi_2 + jb\sin\phi_2 = -643.46\pi - j1554\pi$$

Step 4 : The denominator of $H(s)$ $(s - s_1)(s - s_2)$

$$(s + 643.46\pi + j1554\pi)$$

$$(s + 643.46\pi - j1554\pi)(s + 643.46\pi + j1554\pi)$$

$$(s + 643.46\pi)^2 - (j1554\pi)^2$$

$$(s + 643.46\pi)^2 + (1554\pi)^2$$

Step 5 :

The numerator of $H(s)$

$$\frac{(643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+\varepsilon^2}} = (1414.38)^2\pi^2$$

Step 6 :

The transfer fn
$$H(s) = \frac{(1414.38)^2\pi^2}{s^2 + 1287\pi s + (1682)^2\pi^2}$$

Chebyshev Type - 2 filter

The magnitude square response is given by

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \left[ \dfrac{C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)}{C_N^2\left(\frac{\Omega_s}{\Omega}\right)} \right]}$$

where $C_N(x)$ - $N^{th}$ order chebyshev polynomial.

Order of the filter is given by

$$N \geq \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

The zeros are located on the imaginary axis at the points

$$S_k = j\frac{\Omega_s}{\sin\phi_k} \qquad k = 1, 2, \ldots N.$$

The poles are located at the points $(x_k, y_k)$

where

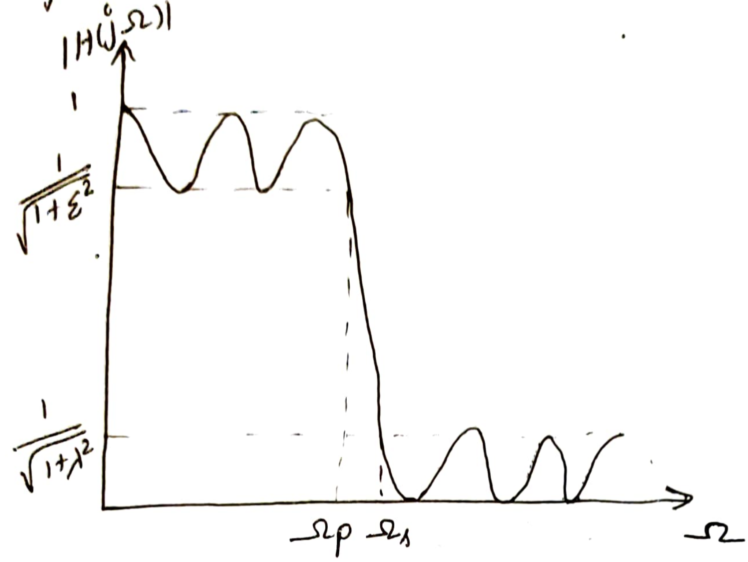
$$x_k = \frac{\Omega_s \sigma_k}{\sigma_k^2 + \Omega_k^2} \qquad k = 1, 2, \ldots N.$$

$$y_k = \frac{\Omega_s \Omega_k}{\sigma_k^2 + \Omega_k^2} \qquad k = 1, 2, \ldots N$$

where $\sigma_k = a\cos\phi_k$

$\Omega_k = b\sin\phi_k$.

## Elliptic filter:




The magnitude square response is given by

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 U_N(\Omega)}$$

where

$U_N(\Omega)$ - Jacobian elliptic function

The poles & zeros are derived from jacobian elliptic sine function. The order of the elliptic filter is given by

$$N \geq \frac{\log 16(A^2)}{\log(1/q)}$$

where

$$A^2 = \frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}$$

$$q = q_0 + 2q_0^5 + 15q_0^9 + 150q_0^{13}$$

$$q_0 = \left[\frac{1-(1-k^2)^{0.25}}{2[1+(1-k^2)^{0.25}]}\right]$$

$$k = \frac{\Omega_p}{\Omega_s}$$

Find the order of elliptic filter for given spec

$\Omega_p = \sqrt{0.9}$ ; $\Omega_s = \frac{1}{\sqrt{0.9}}$ $\alpha_s = 50dB$ ; $\alpha_p = 0.1dB$

Sol

$$k = \frac{\Omega_p}{\Omega_s} = \frac{\sqrt{0.9}}{\frac{1}{\sqrt{0.9}}} = 0.9$$

$$A^2 = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} = \frac{10^{0.1 \times 50} - 1}{10^{0.1 \times 0.1} - 1} = 431030.72$$

$$A = 2071.978$$

$$q_0 = \left[ \frac{1 - (1 - 0.9^2)^{0.25}}{2[1 + (1 - 0.9^2)^{0.25}]} \right]$$

$$= 0.102329$$

$$q = 0.102329 + 2(0.102329)^5 + 15(0.102329)^9 + 150(0.102329)^{13}$$

$$= 0.102351$$

$$N \geq \frac{\log 16(A^2)}{\log(1/q)} = \frac{\log 16(2071.978)^2}{\log(1/0.102351)}$$

$$N \geq 7.916$$

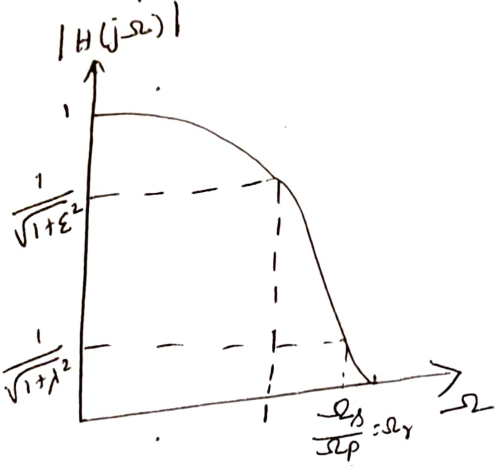
$$N = 8$$

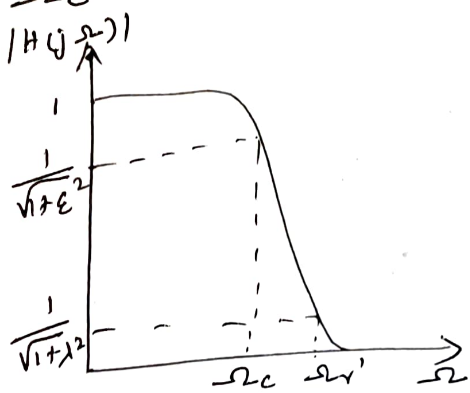
## Frequency Transformation in Analog Domain

It is used to design low pass filters with different pass band frequencies, high pass filters, band pass filters & band stop filters from a normalised low pass analog filter.

### 1. Lowpass to Lowpass filter

A normalized low pass filter into a low pass filter with a different cut off freq. $\Omega_c$ can be accomplished by $s \rightarrow \frac{s}{\Omega_c}$

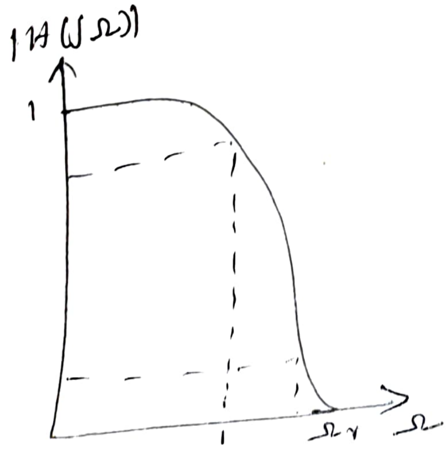


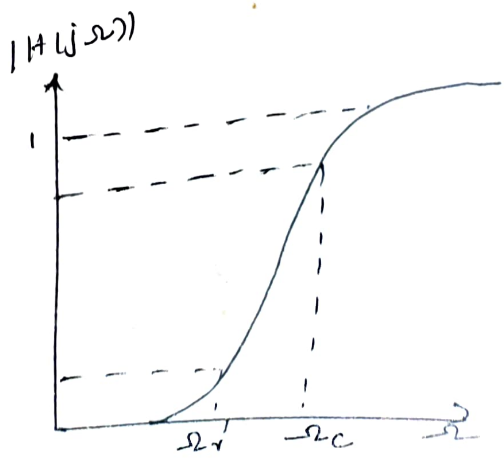




### 2. Lowpass to Highpass

Normalised low pass filter $\longrightarrow$ High pass filter with cut off freq $\Omega_c$

$$s \rightarrow \frac{\Omega_c}{s}$$



3. Lowpass to Bandpass

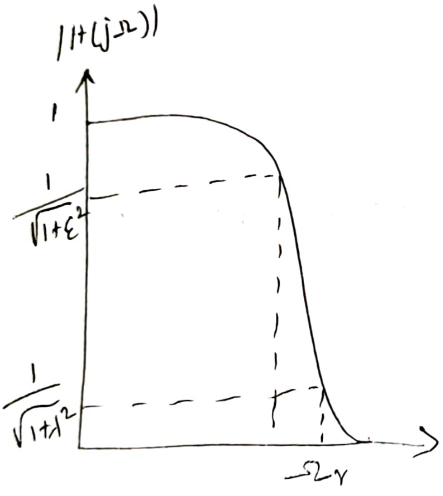
Normalised low pass filter $\rightarrow$ Bandpass filter with cut off frequencies $\Omega_l, \Omega_u$.

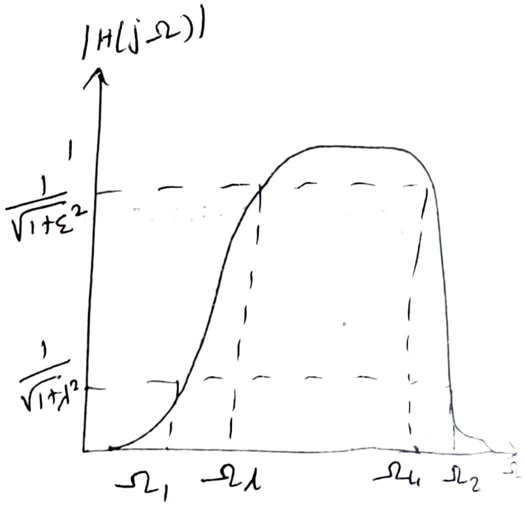
$$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

$$\Omega_r = \min\{|A|, |B|\}$$

$$A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)}$$

$$B = \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$$



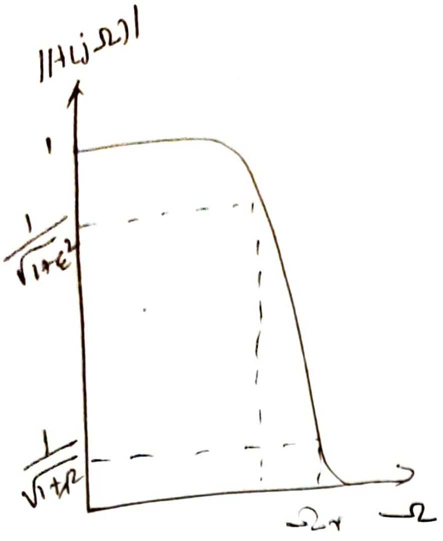

4. Lowpass to Band stop.

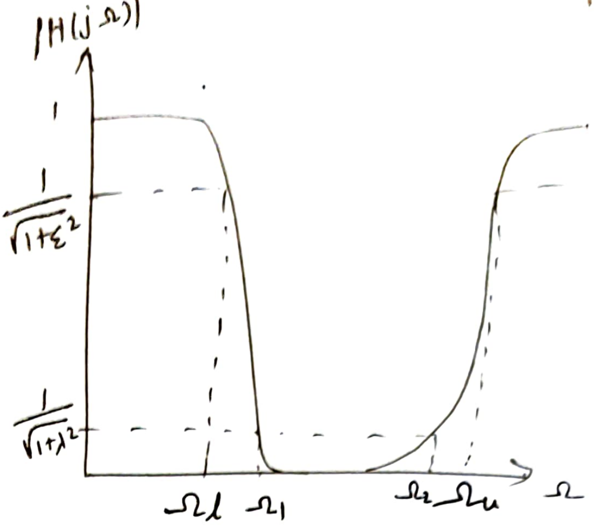
Normalised low pass filter $\rightarrow$ Band stop filter.

$$s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u} \qquad \Omega_r = \frac{\Omega_s}{\Omega_p}$$

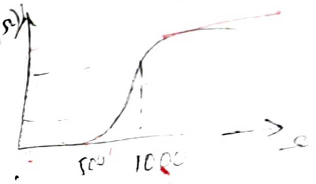
$$\Omega_r = \min\{|A|, |B|\}$$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l \Omega_u} \qquad B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l \Omega_u}$$



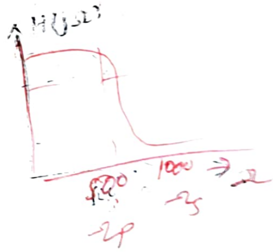




For the given specifications $\alpha_p = 3dB$; $\alpha_s = 15dB$; $\Omega_p = 1000$ rad/sec, & $\Omega_s = 500$ rad/sec design a high pass filter.




sol

Normalised low pass filter design:




$\Omega_p = 500$ rad/sec $\qquad \alpha_p = 3dB$

$\Omega_s = 1000$ rad/sec $\qquad \alpha_s = 15dB.$

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\frac{\Omega_s}{\Omega_p}} = 2.468$$

$\underline{N=3}$

$H(s)$ for $\Omega_c = 1$ rad/sec for $N=3$

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

To get high pass filter with $\Omega_c = \Omega_p = 1000$ r/s

$$s \rightarrow \frac{1000}{s}$$

$$H(s) = \frac{s^3}{(s+1000)(s^2+1000s+1000^2)}$$

Design of IIR filters from analog filters.

Four most widely used methods for digitizing the analog filter

1. Approximation of derivatives
2. The impulse invariant transformation
3. The bilinear transformation
4. The matched z- transfrom technique.

Impulse invariant method:

In impulse invariance method, the IIR filter is designed such that the unit impulse response $h(n)$ of digital filter is the sampled version of the impulse response of analog filter.

$$h(n) = h_a(nT) \quad T - \text{sampling rate}$$

Let $H_a(s)$ is the system function of an analog filter. It can be expressed in partial fraction form as

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s - P_k}.$$

where

$\{P_k\}$ - poles of the analog filter

$\{C_k\}$ - coefficients in the partial fraction expansion

Inverse laplace transform is

$$h_a(t) = \sum_{k=1}^{N} C_k e^{P_k t} \quad t \geq 0$$

If $h_a(t)$ is sampled periodically at $t = nT$

$$h(n) = h_a(nT) = \sum_{k=1}^{N} C_k e^{P_k nT}$$

System function of digital filter can be obtained by taking Z

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \sum_{k=1}^{N} C_k e^{p_k nT} z^{-n}$$

$$= \sum_{k=1}^{N} C_k \sum_{n=0}^{\infty} e^{p_k nT} z^{-n}$$

$$= \sum_{k=1}^{N} C_k \sum_{n=0}^{\infty} \left(e^{p_k T} z^{-1}\right)^n$$

$$\boxed{H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{p_k T} z^{-1}}}$$

ie If $H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s - p_k}$ then $H(z) = \frac{C_k}{1 - e^{p_k T} z^{-1}}$

$$s - p_k = 1 - e^{p_k T} z^{-1}$$

Analog poles are at $s = p_k$ which is mapped into digital poles at $z = e^{p_k T}$

* Note

Design a third order butterworth ~~filter~~ (digital) using impulse invariant technique. Assume sampling period $T = 1 sec$.

Sol

For $N = 3$. the transfer function of a normalised Butterworth filter is given by

$$H(s) = \frac{1}{(s+1)(s^2 + s + 1)}$$

$a = r \quad b = \ldots , \cos$

By taking partial fraction

$$H(s) = \frac{1}{(s+1)(s+0.5+j0.866)(s+0.5-j0.866)}$$

$$= \frac{A}{s+1} + \frac{B}{s+0.5+j0.866} + \frac{C}{s+0.5-j0.866}$$

$$A = (s+1)\left.\frac{1}{(s+1)(s^2+s+1)}\right|_{s=-1}$$

$$= 1$$

$$B = (s+0.5+j0.866)\left.\frac{1}{(s+1)(s+0.5+j0.866)(s+0.5-j0.866)}\right|_{s=-0.5-j0.866}$$

$$= \left.\frac{1}{(s+1)(s+0.5-j0.866)}\right|_{s=-0.5-j0.866}$$

$$= \frac{1}{(-0.5-j0.866+1)(-0.5-j0.866+0.5-j0.866)}$$

$$= \frac{1}{(0.5-j0.866)(-j1.732)}$$

$$= \frac{1}{-j0.866-1.5} =$$

$$= \frac{1}{-1.5-j0.866} \times \frac{-1.5+j0.866}{-1.5+j0.866}$$

$$= \frac{-1.5+j0.866}{3} = -0.5+j0.288$$

$-1.5 - j0.866$

$\times -1.5+j0.866$

$1.5+j0.866$

$-2.3$

$C = B^* = -0.5 - j0.288$

$$H(s) = \frac{1}{s+1} + \frac{-0.5+j0.288}{s+0.5+j0.866} + \frac{-0.5-j0.288}{s+0.5-j0.866}$$

Rearrange $H(s)$ as $\boxed{H(s) = \sum_{k=1}^{N} \frac{C_k}{s-P_k}}$

$$H(s) = \frac{1}{s-(-1)} + \frac{-0.5+j0.288}{s-(-0.5-j0.866)} + \frac{-0.5-j0.288}{s-(-0.5+j0.866)}$$

then $\boxed{H(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{P_kT}z^{-1}}}$

$$H(z) = \frac{1}{1-e^{-T}z^{-1}} + \frac{-0.5+j0.288}{1-e^{(-0.5-j0.866)T}z^{-1}} + \frac{-0.5-j0.288}{1-e^{(-0.5+j0.866)T}z^{-1}}$$

Given that $T=1$.

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} + \frac{-0.5+j0.288}{1-e^{-0.5-j0.866}z^{-1}} + \frac{-0.5-j0.288}{1-e^{-0.5+j0.866}z^{-1}}$$

$$= \frac{1}{1-0.368z^{-1}} + \frac{-1+0.66z^{-1}}{1-0.786z^{-1}+0.368z^{-2}}$$

Note:
For high sampling rates (for small $T$), the digital filter gain is high. Therefore

$$H(z) = \sum_{k=1}^{N} \frac{T\,C_k}{1-e^{P_kT}z^{-1}}$$

Due to the presence of aliasing, the impulse invariant method is appropriate for the design of low pass & bandpass filters only. The impulse invariance method is unsuccessful for implementing digital filter such as highpass filter.

Using impulse invariance with $T = 1$ sec. Determine $H(z)$ if $H(s) = \frac{1}{s^2+\sqrt{2}s+1}$.

sol

Given $H(s) = \frac{1}{s^2+\sqrt{2}s+1}$

$$h(t) = L^{-1}[H(s)] = L^{-1}\left[\frac{1}{s^2+\sqrt{2}s+1}\right]$$

$$= L^{-1}\left[\frac{1}{(s+1/\sqrt{2})^2+(1/\sqrt{2})^2}\right]$$

$$= L^{-1}\left[\sqrt{2}\,\frac{1/\sqrt{2}}{(s+1/\sqrt{2})^2+(1/\sqrt{2})^2}\right]$$

$$= \sqrt{2}\,L^{-1}\left[\frac{1/\sqrt{2}}{(s+1/\sqrt{2})^2+(1/\sqrt{2})^2}\right]$$

$$= \sqrt{2}\,e^{-t/\sqrt{2}}\sin(t/\sqrt{2})$$

Let $t = nT$

$$h(nT) = \sqrt{2}\,e^{-nT/\sqrt{2}}\sin\left(\frac{nT}{\sqrt{2}}\right)$$

If $T = 1$ sec.

$$h(n) = \sqrt{2}\,e^{-n/\sqrt{2}}\sin\left(\frac{n}{\sqrt{2}}\right)$$

$$H(z) = Z\{h(n)\} = \sqrt{2}\left[\frac{e^{-1/\sqrt{2}}z^{-1}\sin(1/\sqrt{2})}{1-2e^{-1/\sqrt{2}}z^{-1}\cos\frac{1}{\sqrt{2}}+e^{-\sqrt{2}}z^{-2}}\right]$$

$$L\left[e^{-bt}\,\text{Sin}\,bt\right] = \frac{b}{(S+a)^2+b^2}$$

For the analog transfer function $H(s) = \frac{2}{(s+1)(s+2)}$ determine $H(z)$ using impulse invariance method. Assume $T = 1$ sec.

sol

Given $H(s) = \frac{2}{(s+1)(s+2)}$

Using partial fraction

$$H(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$A = \cancel{s+1} \frac{2}{(\cancel{s+1})(s+2)} \Big|_{s=-1} = 2$$

$$B = (\cancel{s+2}) \frac{2}{(s+1)(\cancel{s+2})} \Big|_{s=-2} = -2$$

$$\therefore H(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

$s - s_1 \quad s - s_2$

$$= \frac{2}{s-(-1)} - \frac{2}{s-(-2)}$$

Using impulse invariance tech.

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{s-P_k} \quad \text{then} \quad H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

$P_k = -1, -2.$

$$H(z) = \frac{2}{1 - e^{-T} z^{-1}} - \frac{2}{1 - e^{-2T} z^{-1}}$$

Given that $T = 1$ sec

$$H(z) = \frac{2}{1 - e^{-1} z^{-1}} - \frac{2}{1 - e^{-2} z^{-1}} = \frac{0.465 z^{-1}}{1 - 0.503 z^{-1} + 0.0497 z^{-2}}$$

Impulse invariant pole mapping:

Let the infinite impulse response h(n) & its z transform is given by

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$H(z)\Big|_{z=e^{sT}} = \sum_{n=0}^{\infty} h(n)\, e^{-sTn}$$

Let us consider the mapping of points from the s-plane to the z plane implied by the relation

$$z = e^{sT}$$

sub $s = \sigma + j\Omega$ & express the complex variable z in polar form as $z = re^{j\omega}$

$$re^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$= e^{\sigma T} e^{j\Omega T}$$

$$\Rightarrow r = e^{\sigma T} \quad \& \quad \omega = \Omega T$$

$e^{\sigma T}$ – Mag. of $e^{\sigma T}$ & angle 0

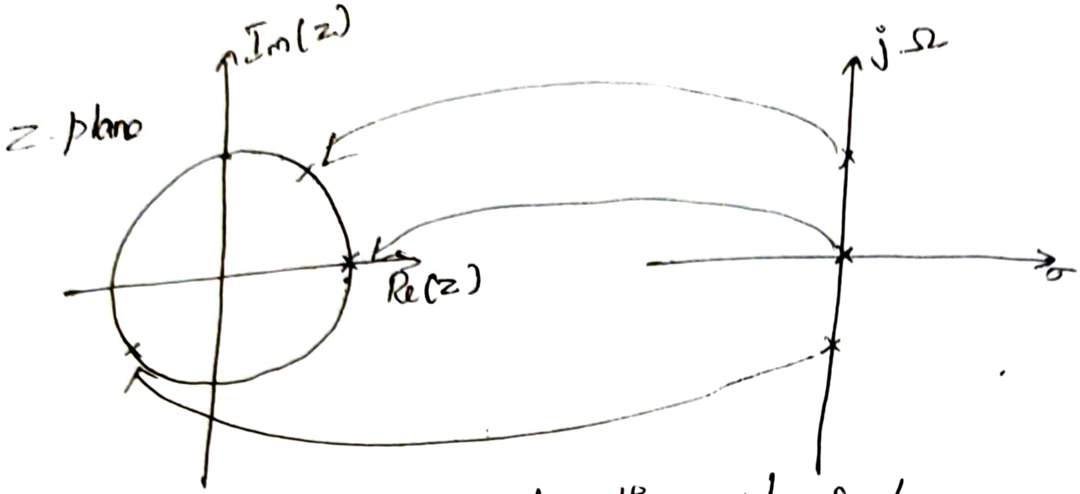
$e^{j\Omega T}$ – Unity mag & an angle $\Omega T$.

∴ Analog pole is mapped to a plane in the z-plane of magnitude $e^{\sigma T}$ & angle $\Omega T$

The real part of the analog pole determines the radius of the z-plane pole & the imaginary part of the analog pole dictates the angle of the digital pole.

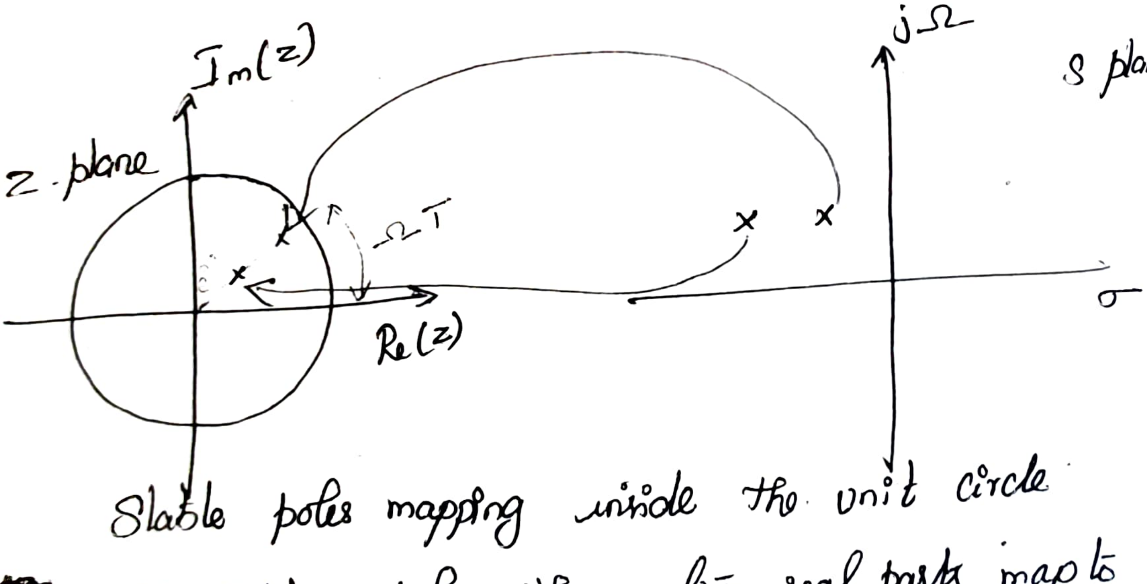
i) Consider any pole on the $j\Omega$ axis where $\sigma = 0$
These poles map to z-plane at a radius

$$r = e^{0 \cdot T} = 1$$




$j\Omega$ axis mapping to the unit circle

ii) Consider the poles in the left half of s-plane where $\sigma < 0$
These poles map inside the unit circle because $r = e^{\sigma T} < 1$ for $\sigma < 0$.

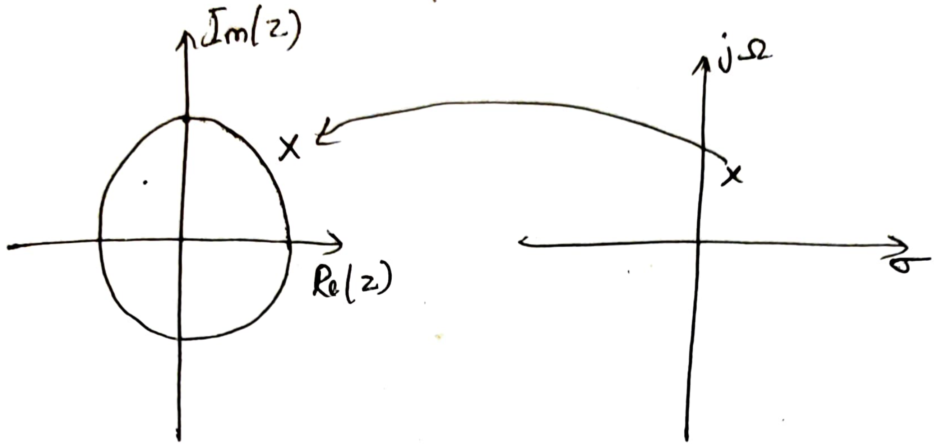



Stable poles mapping inside the unit circle

∴ All s-plane poles with negative real parts map to z-plane poles inside the unit circle - stable analog poles are mapped to stable digital poles. The impulse invariant mapping preserves the stability of the filter

iii) Consider poles in the right half of the s-plane map. These poles map to outside the unit circle.

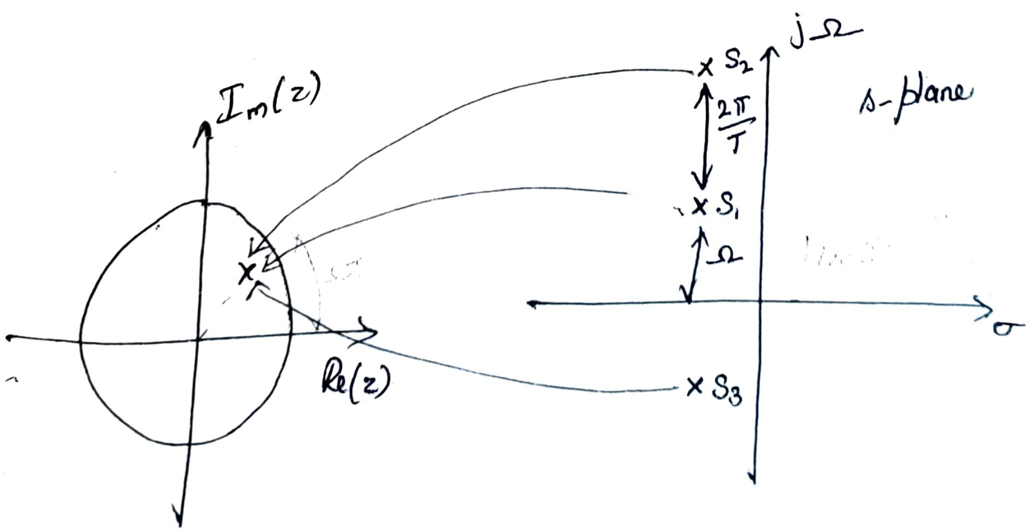
$$r = e^{\sigma T} > 1 \quad \text{for } \sigma > 0.$$




iv) Although the jΩ axis is mapped into the unit circle, it is not one to one mapping rather it is many to one mapping, where many points in s-plane are mapped to a single point in the z-plane.

Consider two poles in the s-plane with identical real parts but with imaginary components differing by $\frac{2\pi}{T}$.

ie $\quad s_1 = \sigma + j\Omega \quad ; \quad s_2 = \sigma + j\left(\Omega + \frac{2\pi}{T}\right)$




These poles map to z-plane poles $z_1$ & $z_2$ via impulse invariant mapping

$$Z_1 = e^{S_1 T} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T}$$

$$Z_2 = e^{S_2 T} = e^{\left[\sigma + j\left(\Omega + \frac{2\pi}{T}\right)\right]T} = e^{\sigma T} \cdot e^{j\Omega T + j2\pi}$$

$$= e^{\sigma T} \cdot e^{j\Omega T} \quad \left(\because e^{j2\pi} = 1\right)$$

∴ These poles map to the same location in the z plane. There are an infinite no. of s-plane poles that map to the same location in the z-plane. They must have the same real parts & imaginary parts that differ by some integer multiple of $\frac{2\pi}{T}$. This is the main disadvantage of impulse invariant mapping.

The s plane poles having imaginary parts greater than $\pi/T$ or less than $-\pi/T$ cause aliasing, when sampling analog signals.

Bilinear Transformation:

- Bilinear transform is a conformal mapping that transforms the $j\Omega$ axis into the unit circle in the z plane only once, thus avoiding aliasing of freq components.

- All RHP of s plane are mapped inside the unit circle in the z-plane & all RHP of s plane are mapped into corresponding points outside the unit circle in the z plane.

Consider analog linear filter system fn

$$H(s) = \frac{b}{s+a} \quad \text{——— (1)}$$

$$\frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$sY(s) + aY(s) = bX(s)$ — (2)

This can be characterized by the difference eq

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \quad \text{— (3)}$$

y(t) can be approximated by the trapezoidal formula

Thus

$$y(t) = \int_{t_0}^{t} y'(t)\,dt + y(t_0) \qquad y'(t) \text{ - derivative of } y(t)$$

approximation at $t = nT$ & $t_0 = nT - T$

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T) \quad \text{— (4)}$$

From (3) $y'(nT)$ is given by

$$y'(nT) = -ay(nT) + bx(nT) \quad \text{— (5)}$$

sub value of (5) in (4)

$$y(nT) = \frac{T}{2}\left[-ay(nT) + bx(nT) - ay(nT-T) + bx(nT-T)\right] + y(nT-T)$$

$$y(nT) + \frac{aT}{2}y(nT) + \frac{aT}{2}y(nT-T) - y(nT-T) = \frac{bT}{2}x(nT) + b\frac{T}{2}x(nT-T).$$

sub $y(nT) = y(n)$ & $x(nT) = x(n)$

$$y(n) + \frac{aT}{2}y(n) + \frac{aT}{2}y(n-1) - y(n-1) = \frac{bT}{2}x(n) + \frac{bT}{2}x(n-1)$$

$$\left(1 + \frac{aT}{2}\right)y(n) - \left(1 - \frac{aT}{2}\right)y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right]$$

By taking z transform

$$\left(1+\frac{aT}{2}\right)Y(z)-\left(1-\frac{aT}{2}\right)z^{-1}Y(z)=\frac{bT}{2}\left[1+z^{-1}\right]X(z)$$

The system fn of the digital filter

$$\frac{Y(z)}{X(z)}=H(z)=\frac{\frac{bT}{2}(1+z^{-1})}{1+\frac{aT}{2}-\left(1-\frac{aT}{2}\right)z^{-1}}$$

$$=\frac{\frac{bT}{2}(1+z^{-1})}{1-z^{-1}+\frac{aT}{2}(1+z^{-1})}$$

Dividing num & den by $\frac{T}{2}(1+z^{-1})$

$$H(z)=\frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a} \quad -⑥$$

Comparing eq ⑥ & ①

$$s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

This relationship b/w s & z is known as bilinear transformation.

Let $z=re^{j\omega}$ & $s=\sigma+j\Omega$

Consider $r=1$ & $\sigma=0$.

$$\cancel{s}+j\Omega=\frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

$$=\frac{2}{T}\left(\frac{e^{j\omega}-1}{e^{j\omega}+1}\right)$$

$$= \frac{2}{T} \frac{\cancel{e^{j\omega/2}}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}{\cancel{e^{j\omega/2}}\left(e^{j\omega/2} + e^{-j\omega/2}\right)}$$

$$= \frac{2}{T} \frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{j\omega/2} + e^{-j\omega/2}}$$

$$= \frac{2}{T} \frac{\frac{e^{j\omega/2} - e^{-j\omega/2}}{2j} \times 2j}{\frac{e^{j\omega/2} + e^{-j\omega/2}}{2} \times 2}$$

$$= \frac{2}{T} \frac{\cancel{2}j \sin \omega/2}{\cancel{2} \cos \omega/2}$$

$$\cancel{j}\Omega = \frac{\cancel{j}2}{T} \tan \omega/2$$

$$\boxed{\Omega = \frac{2}{T} \tan \omega/2}$$

Rearranging for $\omega$

$$\frac{\Omega T}{2} = \tan \omega/2$$

$$\tan^{-1}\left(\frac{\Omega T}{2}\right) = \frac{\omega}{2}$$

$$\boxed{\omega = 2 \tan^{-1}\left(\frac{\Omega T}{2}\right)}$$

The warping effect

Let $\Omega$ & $\omega$ represents frequency variables in analog filter & derived digital filter

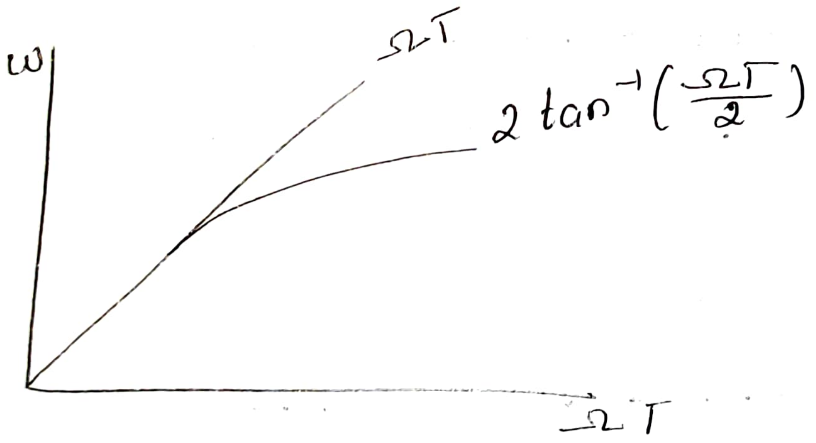
$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

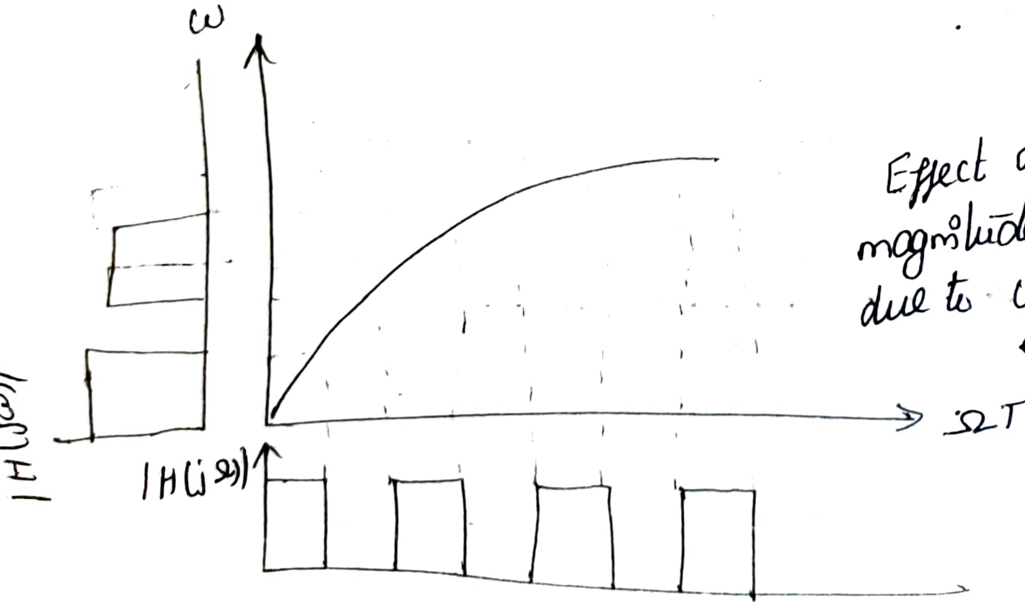
for small values of $\omega$

$$\Omega = \frac{2}{T}\cdot\frac{\omega}{2} = \frac{\omega}{T}$$

$$\therefore \omega = \Omega T$$

- Low freq ⇒ $\omega$ & $\Omega$ – linear ∴ Digital filter have same amplitude response of analog filter.
- High freq ⇒ $\omega$ & $\Omega$ – nonlinear ∴ Distortion is introduced in the frequency scale of the digital filter to that of analog filter. This is known as "Warping Effect."



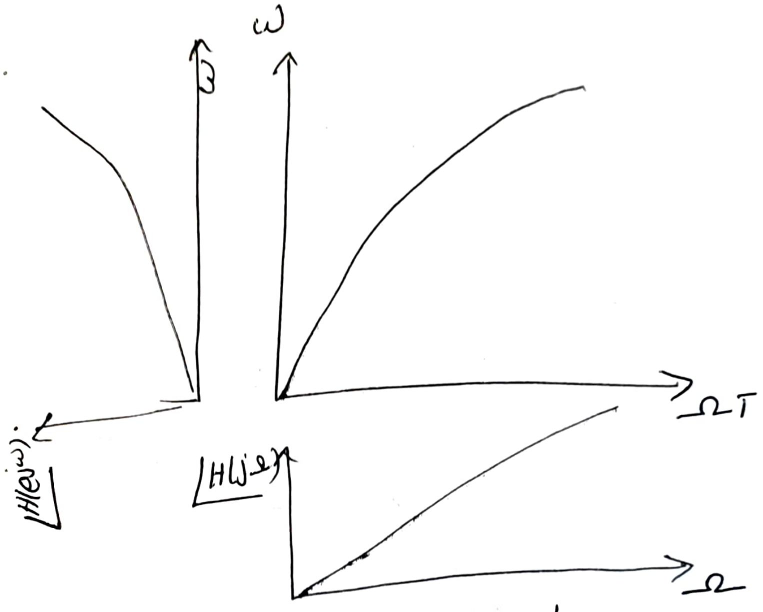

Effect on magnitude response due to warping effect

Prewarping:

The warping effect can be eliminated by prewarping the analog filter. This can be done by finding prewarping analog frequencies using the formula

$$\Omega = \frac{2}{T} \tan\frac{\omega}{2}$$

$\therefore$ $\Omega_p = \frac{2}{T} \tan\frac{\omega_p}{2}$ & $\Omega_s = \frac{2}{T} \tan\frac{\omega_s}{2}$. $\omega = \Omega T$




Effect on phase response due to warping effect

Apply bilinear transformation to $H(s) = \frac{2}{(s+1)(s+2)}$ with $T = 1$ sec & find $H(z)$.

Sol Given $H(s) = \frac{2}{(s+1)(s+2)}$

sub $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ in $H(s)$ to get $H(z)$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Given $\Rightarrow T = 1$ sec

$$H(z) = \frac{2}{\left[2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+1\right]\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right]}$$

$$= \frac{2}{\frac{[2(1-z^{-1})+(1+z^{-1})][2(1-z^{-1})+2(1+z^{-1})]}{(1+z^{-1})^2}}$$

$$= \frac{2(1+z^{-1})^2}{[2-2z^{-1}+1+z^{-1}][2-2z^{-1}+2+2z^{-1}]}$$

$$= \frac{2(1+z^{-1})^2}{(3-z^{-1})4}$$

$$= \frac{(1+z^{-1})^2}{6-2z^{-1}}$$

$$= \frac{0.166(1+z^{-1})^2}{1-0.83z^{-1}}$$

Using the bilinear transform, design a high pass filter, Butterworth monotonic in pass band with cut off freq of 1000Hz & attenuation of 3dB & down 10dB at 350Hz. The sampling freq is 5000Hz.

Sol

Given $\alpha_p = 3dB$; $\omega_c = \omega_p = 2\pi \times 1000 = 2000\pi$ rad/sec.

$\alpha_s = 10dB$; $\omega_s = 2\pi \times 350 = 700\pi$ rad/sec.

$T = \frac{1}{f} = \frac{1}{5000} = 2 \times 10^{-4}$ sec.

The characteristics are monotonic in both passband & stop band. ∴ the filter is butterworth filter.

Step 1: Prewarping analog freq.

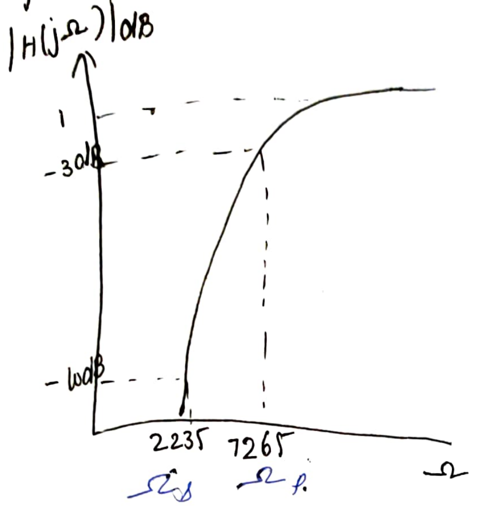
$\boxed{\Omega_p = \frac{2}{T} \tan \frac{\omega_p T}{2}} = \frac{2}{2 \times 10^{-4}} \tan\left(\frac{2000\pi \times 2 \times 10^{-4}}{2}\right)$

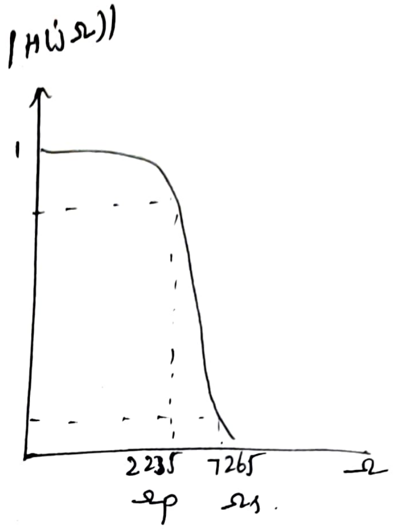
$= 7265$ rad/sec

$$\boxed{\Omega_s = \frac{2}{T}\tan\frac{\omega_s T}{2}} = \frac{2}{2\times10^{-4}}\tan\left(\frac{700\pi\times2\times10^{-4}}{2}\right)$$

$$= 2235 \text{ rad/sec.}$$

Step 2: Analog filter transfer fn $H(s)$.




To design analog low pass filter $\Omega_p = 2235$ rad/sec.
$\alpha_p = 3dB$ & $\Omega_s = 7265$ & $\alpha_s = 10dB$.

$$N \geq \frac{\log\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}} = 0.932 \qquad N = 1.$$

Normalised transfer fn of LP first order filter $\Big\}$ $H(s) = \dfrac{1}{s+1}$

High pass filter for $\Omega_c = \Omega_p = 7265$ can be obtained by using the transformation $s \rightarrow \dfrac{\Omega_c}{s} = \dfrac{7265}{s}$

$\therefore$ The transfer fn of high pass filter $= \dfrac{1}{\frac{7265}{s}+1}$

$$= \frac{s}{s+7256}$$

Step 3: Digital filter tr. fn. Using bilinear transformation

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$T = 2\times10^{-4}$ sec

$$H(z) = \frac{\frac{2}{2\times10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{\frac{2}{2\times10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 7265}$$

$$= \frac{0.5792\,(1-z^{-1})}{1-0.1584\,z^{-1}}$$

Determine the system function $H(z)$ of the lowest order chebyshev & butterworth digital filter with the following specifications

a) 3dB ripple in pass band $0 \le \omega \le 0.2\pi$

b) 25dB attenuation in stop band $0.45\pi \le \omega \le \pi$

Sol

Butterworth Filter

Given $\omega_p = 0.2\pi$ $\quad \omega_s = 0.45\pi$ $\quad \alpha_p = 3dB$ & $\alpha_s = 25dB$

$T = 1$ sec.

Step 1: Prewarping

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 2\tan\left(\frac{0.2\pi}{2}\right) = 0.65.$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = 2\tan\left(\frac{0.45\pi}{2}\right) = 1.71$$

Step 2; Analog filter design

$$N \ge \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\frac{\Omega_s}{\Omega_p}} = 2.97 \qquad N = 3$$

Normalised low pass filter for order $N=3$ $\Big\}$ $H(s) = \dfrac{1}{(s+1)(s^2+s+1)}$

To obtain the tr. fn for $\Omega_c = 0.65$ sub

$$s \to \frac{s}{0.65}$$

$T = 2\times10^{-4}$ sec

$$H(z) = \frac{\frac{2}{2\times10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{\frac{2}{2\times10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 7265}$$

$$= \frac{0.5792\,(1-z^{-1})}{1-0.1584\,z^{-1}}$$

Determine the system function $H(z)$ of the lowest order chebyshev & butterworth digital filter with the following specifications

a) 3dB ripple in pass band $0 \le \omega \le 0.2\pi$

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sol

Butterworth Filter

Given $\omega_p = 0.2\pi \quad \omega_s = 0.45\pi \quad \alpha_p = 3dB \;\&\; \alpha_s = 25dB$

$T = 1$ sec.

Step 1: Prewarping

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 2\tan\left(\frac{0.2\pi}{2}\right) = 0.65.$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = 2\tan\left(\frac{0.45\pi}{2}\right) = 1.71$$

Step 2; Analog filter design

$$N \ge \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\frac{\Omega_s}{\Omega_p}} = 2.97 \qquad N = 3$$

Normalised low pass filter for order $N=3$ $\Big\}$ $H(s) = \dfrac{1}{(s+1)(s^2+s+1)}$

To obtain the tr. fn for $\Omega_c = 0.65$ sub

$$s \to \frac{s}{0.65}$$

$$H(s) = \frac{1}{\left(\frac{s}{0.65}+1\right)\left[\left(\frac{s}{0.65}\right)^2 + \left(\frac{s}{0.65}\right)+1\right]}$$

$$= \frac{0.65^3}{(s+0.65)(s^2+0.65s+0.4225)}$$

Step 3: Digital filter tr. Using bilinear transformation

$$s \rightarrow \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{0.02066\,(1+z^{-1})^3}{(1-0.51z^{-1})(1-1.41z^{-1}+0.751z^{-2})}$$

Chebyshev filter:

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}} \qquad N = 3.$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p}-1} = 1$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 2.414$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.A35$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{1/N}}{2}\right] = 0.678$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)}{2N} \quad ; \; k = 1, 2, 3$$

$$\phi_1 = 120 \quad \phi_2 = 180 \quad \phi_3 = 240$$

$$S_1 = a\cos\phi_1 + jb\sin\phi_1 = -0.09675 + j0.587$$

$$S_2 = a\cos\phi_2 + jb\sin\phi_2 = -0.1935$$

$$S_3 = a\cos\phi_3 + jb\sin\phi_3 = -0.09675 - j0.587$$

The denominator poly of $H(s)$ $= (s+0.1935)\left[(s+0.09675)^2 + 0.587^2\right]$

The numerator poly obtained by sub $s=0$ in den. since $N = 3$ (odd)

Num. poly $= (0.1935)(0.354) = 0.0685$

The tr. fn $= \dfrac{0.0685}{(s+0.1935)\left[(s+0.09675)^2 + (0.587^2)\right]}$

The tr. fn of digital filter can be obtained by Using bilinear transformation by

$$s \rightarrow \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{0.00687(1+z^{-1})^3}{(1-0.823z^{-1})(1-1.6z^{-1}+0.915z^{-2})}$$

The fourier series method of designing FIR filters.

Frequency response $H(e^{j\omega})$ is periodic in $2\pi$. Then any periodic function can be expressed as a linear combination of complex exponentials.

The desired freq. response of an FIR filter can be represented by the fourier series,

$$\boxed{H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}}$$

where

$h_d(n)$ – fourier coefficients which represents the desired impulse sequence of the filter.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega.$$

To obtain the tr. fn of the filter apply Z-transform of the sequence $h_d(n)$

$$H(z) = \sum_{n=-\infty}^{\infty} h_d(n) z^{-n}$$

Represents a non-causal digital filter of infinite duration. To get an FIR filter tr. fn, the series can be truncated by assigning

$$h(n) = h_d(n) \quad \text{for } |n| \le \frac{N-1}{2}$$
$$= 0 \quad \text{otherwise.}$$

Then $$H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) z^{-n}$$

$$= h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \ldots + h(1)z^{-1} + h(0) + h(-1)z^{1} + \ldots + h\left[-\left(\frac{N-1}{2}\right)\right] z^{\left(\frac{N-1}{2}\right)}$$

For a symmetrical impulse response having symmetry at $n=0$ $h(-n) = h(n)$.

$\therefore$ $H(z)$ can be written as

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)[z^n + z^{-n}]$$

The above tr. fn. is not physically realizable. Realizability can be brought by multiplying by $z^{-\left(\frac{N-1}{2}\right)}$ where $\left(\frac{N-1}{2}\right)$ is delay in samples.

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} \cdot H(z)$$

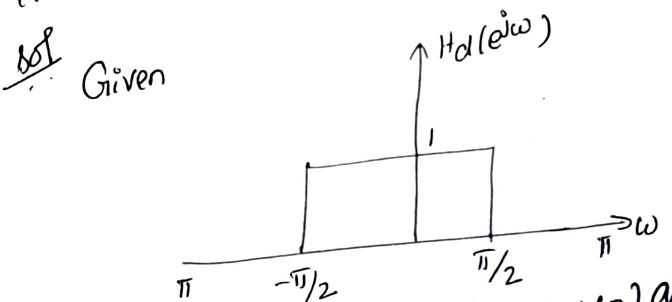
$$= z^{-\left(\frac{N-1}{2}\right)}\left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)[z^n + z^{-n}]\right]$$

Design an ideal low pass filter with a freq. response

$$H_d(e^{j\omega}) = 1 \quad \text{for } -\pi/2 \le \omega \le \pi/2$$
$$= 0 \quad \pi/2 \le |\omega| \le \pi$$

Find the values of $h(n)$ for $N = 11$. Find $H(z)$.

Sol: Given




The ~~filter~~ Fourier coefficients ~~are~~ ~~be~~ $h_d(n)$ are the desired impulse response sequence of the filter $\phi$ is given by.

$$\therefore h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/2}^{\pi/2}$$

$e^{ja} = \cos a + j\sin a$
$e^{-ja} = \cos a - j\sin a$

$$= \frac{1}{2\pi j n}\left[e^{j\pi n/2} - e^{-j\pi n/2}\right]$$

$\cos \pi n/2 + j\sin \pi n/2 - \cos \pi n/2 + j\sin \pi n/2$

$$= \frac{1}{\pi n}\sin \frac{\pi}{2}n \qquad -\alpha \le n \le \alpha$$

Since $N = 11$. Truncating $h_d(n)$ to 11 samples

$$h(n) = \frac{\sin \frac{\pi}{2}n}{\pi n} \quad \text{for} \quad |n| \le 5$$

For $n = 0$.

$$h(0) = \lim_{n\to 0} \frac{\sin \frac{\pi}{2}n}{\pi n} = \frac{1}{2}\lim_{n\to 0}\frac{\sin \frac{\pi}{2}n}{\frac{\pi}{2}n}$$

$$= \frac{1}{2} \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right]$$

[Directly sub. $n=0$ eq becomes indeterminate so initial value theorem]

$n=1$

$$h(1) = h(-1) = \frac{\sin \pi/2}{\pi} = \frac{1}{\pi} = 0.3183$$

$h(2) = h(-2) = 0$ $\qquad$ $h(4) = h(-4) = 0$

$h(3) = h(-3) = -0.106$ $\qquad$ $h(5) = h(-5) = 0.06366$

The transfer fn of the filter is given by

$$\boxed{H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}}\left[h(n)(z^n + z^{-n})\right]}$$

$$= 0.5 + \sum_{n=1}^{5} h(n)(z^n + z^{-n})$$

$$= 0.5 + 0.3183(z^1 + z^{-1}) - 0.106(z^3 + z^{-3}) + 0.06366(z^5 + z^{-5})$$

The tr. fn. of the realizable filter is

$$\boxed{H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)}$$

$$= z^{-5}\left[0.5 + \cdots\cdots\right]\cdots$$

$$= 0.06366 - 0.106z^{-2} + 0.3183z^{-4} + 0.5z^{-5} + 0.3183z^{-6} - 0.106z^{-8} + 0.06366z^{-10}$$

$h(0) = h(10) = 0.06366$; $h(1) = h(9) = 0$; $h(2) = h(8) = -0.106$

... $h(5) = 0.5$

Design of FIR filters using windows:

The abrupt truncation of the fourier series results in oscillation in the pass band & stop band. These oscillations are due to slow convergence of the fourier series & this effect is known as the Gibbs phenomenon.

To reduce these oscillations, the fourier coefficients of the filter are modified by multiplying the infinite impulse response with a finite weighing sequence $w(n)$ called a window where

$$w(n) = w(-n) \neq 0 \qquad |n| \leq \left(\frac{N-1}{2}\right)$$
$$= 0 \qquad |n| > \left(\frac{N-1}{2}\right)$$

After multiplying window sequence $w(n)$ with $h_d(n)$, a finite duration sequence $h(n)$ that satisfies the desired mag response

$$h(n) = h_d(n)\,w(n) \quad \text{for all } |n| \leq \left(\frac{N-1}{2}\right)$$
$$= 0 \quad \text{for } |n| > \left(\frac{N-1}{2}\right)$$

1. Rectangular Window.

Rectangular window sequence is given by

$$W_R(n) = 1 \quad \text{for } -\left(\frac{N-1}{2}\right) \leq n \leq \left(\frac{N-1}{2}\right)$$
$$= 0 \quad \text{otherwise.}$$

2. Hanning Window

Hanning window sequence is given by

$$W_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1} \quad \text{for } -\left(\frac{N-1}{2}\right) \leq n \leq \left(\frac{N-1}{2}\right)$$
otherwise.

3. Hamming Window

Hamming window sequence is given by

$$W_H(n) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right) \text{ for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right)$$

$$= 0 \qquad \text{otherwise}$$

4. Kaiser Window

Kaiser window sequence is given by

$$W_K(n) = \frac{I_0\left[\alpha\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right]}{I_0(\alpha)} \quad \text{for } |n| \le \left(\frac{N-1}{2}\right)$$

$$= 0 \qquad \text{otherwise.}$$

where $\alpha$ - adjustable parameter

$I_0(x)$ - modified zeroth - order Bessel fn of the first kind

$$I_0(x) = 1 + \sum_{k=1}^{\alpha}\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2$$

$$= 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^2} + \dots$$

Design an ideal high pass filter with a freq. response

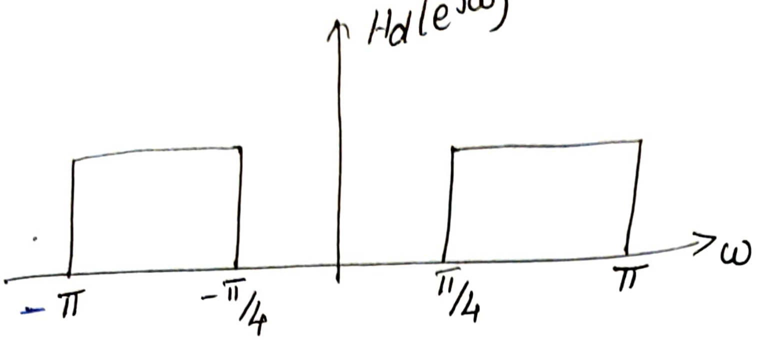
$$H_d(e^{j\omega}) = 1 \quad \text{for } \pi/4 \le |\omega| \le \pi$$

$$= 0 \quad \text{for } |\omega| \le \pi.$$

find the values of $h(n)$ for $N=11$ using Hanning window.

Sol

Ideal frequency response.




1. $$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/4} e^{j\omega n}\, d\omega + \int_{\pi/4}^{\pi} e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\pi/4} + \left[\frac{e^{j\omega n}}{jn}\right]_{\pi/4}^{\pi}\right\}$$

$$= \frac{1}{2\pi j n}\left[e^{-j\frac{\pi n}{4}} - e^{-j\pi n} + e^{j\pi n} - e^{j\pi/4\, n}\right]$$

$$= \frac{1}{\pi n}\left[\sin \pi n - \sin \tfrac{\pi}{4} n\right] \quad -\infty \le n \le \infty$$

2. Hanning window

$$\omega_{Hn}(n) = 0.5 + 0.5\cos\left(\frac{2\pi n}{N-1}\right) \quad \text{for } |n| \le \frac{N-1}{2}$$
$$= 0 \quad \text{otherwise.}$$

for $N = 11$.

$$\omega_{Hn}(n) = 0.5 + 0.5\cos\left(\frac{2\pi n}{10}\right) \quad -5 \le n \le 5$$
$$= 0 \quad \text{otherwise.}$$

$$\omega_{Hn}(n) = 0.5 + 0.5\cos\left(\frac{\pi n}{5}\right) \quad -5 \le n \le 5$$

$W_{Hn}(0) = 0.5 + 0.5 = 1$

$W_{Hn}(1) = W_{Hn}(-1) = 0.5 + 0.5 \cos \pi/5 = 0.9045$

$W_{Hn}(2) = W_{Hn}(-2) = 0.5 + 0.5 \cos(2\pi/5) = 0.655$

$W_{Hn}(3) = W_{Hn}(-3) = 0.345$

$W_{Hn}(4) = W_{Hn}(-4) = 0.0945$

$W_{Hn}(5) = W_{Hn}(-5) = 0.$

3. The filter coefficients for $N = 11$

$$hd(n) = \frac{\sin \pi n - \sin \frac{\pi}{4} n}{\pi n}$$

$$hd(0) = \lim_{n \to 0} \left[ \frac{\sin \pi n - \sin \frac{\pi}{4} n}{\pi n} \right] = 1 - \frac{1}{4}$$

$$= 0.75$$

$$= \lim_{n \to 0} \frac{\sin \pi n}{\pi n} - \lim_{n \to 0} \frac{\sin \frac{\pi}{4} n}{\frac{\pi n}{4}} \cdot \frac{1}{4} = 1 - 0.25$$

$$= 0.75$$

$$hd(1) = hd(-1) = \frac{\sin \pi - \sin \pi/4}{\pi} = -0.225$$

$$hd(2) = hd(-2) = \frac{\sin 2\pi - \sin \pi/2}{2\pi} = -0.159$$

$hd(3) = hd(-3) = -0.075$

$hd(4) = hd(-4) = 0$

$hd(5) = hd(-5) = 0.045$

4. The filter coefficients using hanning window

$h(n) = hd(n) W_{Hn}(n)$ for $|n| \le 5$

$= 0$ otherwise

$h(0) = h_d(0)\omega_{Hn}(0) = 0.75$

$h(-1) = h(1) = h_d(1)\,\omega_{Hn}(1) = -0.204$

$h(-2) = h(2) = \quad = -0.104$

$h(-3) = h(3) = \quad = -0.026$

$h(-4) = h(4) = \quad = 0$

$h(-5) = h(5) = \quad = 0.$

6. The transfer fn of the filter is given by

$$\boxed{H(z) = h(0) + \sum_{n=1}^{5} h(n)\left[z^{n} + z^{-n}\right]}$$

$$= 0.75 - 0.204\,(z + z^{-1}) - 0.104\,(z^{2} + z^{-2})$$
$$- 0.026\,(z^{3} + z^{-3}).$$

7. The transfer fn of the realizable filter

$$H'(z) = z^{-5} H(z).$$
$$= -0.026z^{-2} - 0.104z^{-3} - 0.204z^{-4} + 0.75z^{-5} - 0.204z^{-6} - 0.104z^{-7} - 0.026z^{-8}$$

8. The causal filter coefficients are

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{N-1/2} a(n)\cos\omega n$$

where $a(0) = h\left(\frac{N-1}{2}\right)$

$a(n) = 2h\left(\frac{N-1}{2} - n\right)$

$h(0) = h(1) = h(9)$
$= h(10) = 0$
$h(2) = h(8) = -0.026$
$h(3) = h(7) = -0.104$
$h(4) = h(6) = -0.204$
$h(5) = 0.75$

$a(0) = h(5) = 0.75$

$a(1) = 2h(4) = -0.408$

$a(2) = 2h(3) = -0.208$

$a(3) = 2h(2) = -0.052$

$a(4) = 2h(1) = 0$

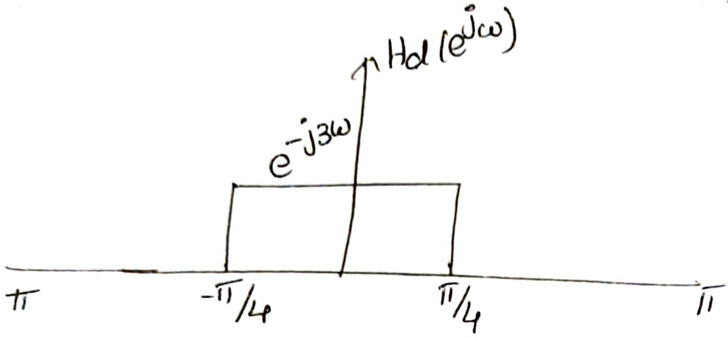
$a(5) = 2h(0) = 0.$

$\bar{H}(e^{j\omega}) = 0.75 - 0.408\cos\omega - 0.208\cos 2\omega - 0.052$

Design a filter with

$$H_d(e^{j\omega}) = e^{-j3\omega} \qquad -\pi/4 \le \omega \le \pi/4$$
$$= 0 \qquad \pi/4 < |\omega| \le \pi$$

Using a Hamming window with $N = 7$




$N = 7$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{-j3\omega} e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{j(n-3)\omega} \, d\omega$$

$$= \frac{\sin \pi/4 \,(n-3)}{\pi (n-3)}$$

$n = \frac{N-1}{2} = 3$ . The freq response is having a term $e^{-j\omega\left(\frac{N-1}{2}\right)}$ ∴ $h(n)$ symmetrical about $n = 3$.

$h_d(0) = h_d(6) = 0.075$

$h_d(1) = h_d(5) = 0.159$

$h_d(2) = h_d(4) = 0.22$

$h_d(3) = 0.25$

1. Determine if the system described by the following i/p-o/p equation is linear or non linear

i) $y(n) = x(n) + \frac{1}{x(n-1)}$  ii) $y(n) = n x^2(n)$.

2. Determine if the following systems are time invariant

i) $y(n) = x(n/2)$  ii) $y(n) = n x^2(n)$.

3. Test the stability of the system whose impulse response

i) $h(n) = (1/2)^n u(n)$  ii) $h(n) = \sin \frac{n\pi}{2}$.  iii) $h(n) = 2^n u(-n)$

4. Determine the frequency response, magnitude response, phase response for the system given by

$$y(n) - \tfrac{3}{4} y(n-1) + \tfrac{1}{8} y(n-2) = x(n) - x(n-1).$$

5. Find the fourier transform of the following.

i) $\delta(n)$  ii) $u(n)$  iii) $a^n u(n)$  iv) $\delta(n+2) - \delta(n-2)$

6. Determine the impulse response $h(n)$ for the system described by the second order difference eq

$$y(n) = 0.6 y(n-1) - 0.08 y(n-2) + x(n)$$

7. Find the Z - transform of the sequence

i) $x(n) = \cos n\theta \, u(n)$  ii) $x(n) = \left(\tfrac{1}{3}\right)^{n-1} u(n-1)$

iii) $x(n) = \tfrac{1}{2}\delta(n) + \delta(n-1) - \tfrac{1}{3}\delta(n-2)$

8. Obtain the direct form I, direct form II, transposed direct form II, cascade form & parallel.

$$y(n) = y(n-1) - \tfrac{1}{2} y(n-2) + \tfrac{1}{4} y(n-3) + x(n) - x(n-1) + x(n-2)$$

Find z transform

$x(n) = (\sin\omega_0 n)\, u(n)$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sin\omega_0 n\, u(n)\, z^{-n}.$$

$$= \sum_{n=0}^{\infty} \sin\omega_0 n\, z^{-n}.$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right] z^{-n}$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} \left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right) z^{-n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} e^{j\omega_0 n} z^{-n} - \sum_{n=0}^{\infty} e^{-j\omega_0 n} z^{-n}\right]$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} \left(e^{j\omega_0} z^{-1}\right)^n - \sum_{n=0}^{\infty} \left(e^{-j\omega_0} z^{-1}\right)^n\right]$$

$$= \frac{1}{2j}\left[\frac{1}{1 - e^{j\omega_0} z^{-1}} - \frac{1}{1 - e^{-j\omega_0} z^{-1}}\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{1} - e^{-j\omega_0} z^{-1} - \cancel{1} + e^{j\omega_0} z^{-1}}{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}\right]$$

$$= \frac{(\sin\omega_0) z^{-1}}{1 - 2(\cos\omega_0) z^{-1} + z^{-2}}$$

**ROC:** Since z transform is an infinite power series, it exists only for those values of z for which this series converges.

The ROC of $X(z)$ is the set of all values of $z$ for which $X(z)$ attains a finite value.

Let us express z in polar form

$$z = re^{j\theta}$$

where $|z| = r$ & $\angle z = \theta$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n)\, r^{-n} e^{-j\theta n}$$

In the ~~the~~ ROC of $X(z)$, $|X(z)| < \infty$ but

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n)\, r^{-n} e^{-j\theta n} \right|$$

$$\leq \sum_{n=-\infty}^{\infty} \left| x(n)\, r^{-n} e^{-j\theta n} \right| = \sum_{n=-\infty}^{\infty} \left| x(n)\, r^{-n} \right|$$

$\therefore$ Finding ROC for $X(z)$ is equivalent to determining the range of values of $r$ for which the sequence $x(n)\, r^{-n}$ is absolutely summable.